

Minimum Variance, Maximum Diversification, and Risk Parity: An Analytic Perspective

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Abstract

Analytic solutions to Minimum Variance, Maximum Diversification, and Risk Parity portfolios provide helpful intuition about their properties and construction. Individual asset weights depend on systematic and idiosyncratic risk in all three risk-based portfolios, but systematic risk eliminates many investable assets in long-only constrained Minimum Variance and Maximum Diversification portfolios. On the other hand, all investable assets are included in Risk Parity portfolios, and idiosyncratic risk has little impact on the magnitude of the weights. The algebraic forms for optimal asset weights derived in this paper provide generalizable perspectives on risk-based portfolio construction, in contrast to empirical simulations that employ a specific set of historical returns, proprietary risk models, and multiple constraints.

Minimum Variance, Maximum Diversification, and Risk Parity: An Analytic Perspective

Portfolio construction techniques based on predicted risk, without expected returns, have become popular in the last decade. In terms of asset selection, Minimum Variance and more recently Maximum Diversification objective functions have been explored, motivated in part by the cross-sectional equity risk anomaly. Application of these objective functions to large (e.g., 1000 stock) investable sets requires sophisticated estimation techniques for the risk model. On the other end of the spectrum, the principal of Risk Parity, traditionally applied to small-set (e.g., 2 to 10) asset allocation decisions, has been proposed for large-set security selection applications. Unfortunately, most of the published research on these strategies is based on standard unconstrained portfolio theory, matched with long-only simulations. The empirical results in such studies are specific to the investable set, time period, maximum weight limits and other portfolio constraints, as well as the risk model. The purpose of this paper is to compare and contrast risk-based portfolio construction techniques using long-only analytic solutions. We also provide a simulation of the risk-based portfolios for large-cap U.S. stocks using the CRSP database from 1968 to 2010. The simulation is performed using a single-index model, standard OLS risk estimates, and without maximum position or other portfolio constraints, leading to easily replicable and unbiased results.

As reviewed in Chow, Hsu, Kalesnik, and Little (2011), Minimum Variance portfolios have been defined and analyzed from the start of modern portfolio theory (i.e., 1960s) as a special case of mean-variance efficient portfolios. The objective function is minimization of ex-ante portfolio risk so that the Minimum Variance portfolio lies on the left-most tip of the efficient frontier. Maximum Diversification portfolios are based on a more recently introduced objective function by Choueifaty and Coignard (2008) that maximizes the ratio of weighted-average asset volatilities to portfolio volatility, an objective that is similar to maximizing the Sharpe ratio, but with asset volatilities replacing asset expected returns.

The concept of Risk Parity has evolved over time from the original concept embedded in research by Bridgewater in the 1990's. Initially, an asset allocation portfolio was said to be in "risk parity" when weights are proportional to asset-class inverse volatility. For example, if an equity portfolio has forecasted volatility of 15%, and a fixed-income portfolio has a volatility of just 5%, then a combined portfolio of 75 percent fixed-income and 25 percent equity (i.e., three times as much fixed-income) is said to be in Risk Parity. The early definition of Risk Parity ignored correlations even as the concept was applied to more than two asset classes. A more complete definition, which considers correlations, was formalized by Qian (2006) who couched the property in terms of a risk budget where weights are adjusted so that each asset has the same contribution to portfolio risk. Maillard, Roncalli, and Teiletche (2010) call this an "equal risk contribution" portfolio, and analyzed properties of an unconstrained analytic solution. An equivalent "portfolio beta" interpretation by Lee (2011) is that Risk Parity is achieved when weights are proportional to the inverse of their beta with respect to the final portfolio. Risk Parity equalizes the *total* risk contribution of each asset to the portfolio, while Minimum Variance and Maximum Diversification portfolios equalize *marginal* contributions given a small change in an asset weight. Risk Parity is not an objective function that can be optimized and has been numerically difficult to implement on large-scale investable sets. Our new analytic solution allows for quick calculation of Risk Parity weights on any sized set for a general K -factor risk model.

We first summarize how the position of the risk-based portfolios in traditional risk-return charts depends on one's implicit assumption about the cross-sectional correspondence between asset risk and return. Under the assumption that forecasted returns are perfectly proportional to risk, the Maximum Diversification portfolio dominates the other two risk-based portfolios in terms of expected Sharpe ratio. On the other hand, if forecasted returns are only slightly increasing in risk, the Minimum Variance portfolio dominates ex-ante, followed by Risk Parity. If returns are assumed to be the same for all stocks, or simply not forecasted, the Minimum Variance portfolio dominates by definition as the lowest ex-ante risk portfolio. If returns are forecasted to be lower for higher risk assets, the Maximum Diversification portfolio is actually on the "wrong side" of the Efficient Frontier. We briefly document the now well-known fact that while higher risk asset classes (e.g., equity compared to fixed-income) have higher long-

term realized returns, the cross-section of stocks has *not* provided a within-equity risk-premium over time.

We then review analytic solutions for long-only constrained risk-based portfolios under the simplifying assumption of a single-factor risk model, with derivations provided in the technical appendix. We explain how the analytic solutions provide intuition for the comparative weight structure of Minimum Variance, Maximum Diversification, and Risk Parity portfolios. Asset weights are shown to be decreasing in both systematic and idiosyncratic risk for all three portfolios, although the form and impact of these two sources of risk vary. For example, Minimum Variance weights are generally proportional to inverse variance (standard deviation squared), while Maximum Diversification and Risk Parity weights are generally proportional to inverse volatility (standard deviation). The analytic solutions reveal why long-only Minimum Variance and Maximum Diversification portfolios employ a relatively small portion (e.g., 100 out of 1000) of the investable set, in contrast to Risk Parity portfolios which include all investable assets, and specify the criterion for inclusion in the long-only solutions. Despite having the highest weight concentration, Minimum Variance portfolios have the lowest ex-ante risk by design, as well as the lowest risk ex-post depending on the accuracy of the risk forecasts. The analytic solutions also provide simple numerical recipes for risk-based portfolio construction in large investable sets, including Risk Parity which heretofore has been a particularly complex optimization problem, as explained in Maillard, Roncalli, and Teiletche (2010). We focus on the intuition provided by a single-factor risk model, but provide solutions for a general K -factor risk model in the technical appendix.

Expected Return and Risk-Based Portfolios

Figures 1 to 3 present traditional risk-return charts for several assets along with the position of the three portfolios under consideration. The focus of this study is on risk-based portfolio construction methodologies that do not employ expected returns, but traditional charts require some assumption in order to map out the mean-variance efficient frontier. In addition, the portfolios in Figures 1 to 3 are unconstrained in order to display the familiar hyperbolic shape of the efficient frontier curve, although we subsequently focus on the analytic form of long-only constrained solutions. Finally, the figures are drawn with respect to returns in excess of the risk-free rate, so the tangency line starts at the origin rather than further up the vertical axis.

Throughout this paper, all returns, either expected or realized, will be stated in excess of the contemporaneous risk-free rate as measured by one-month T-bill returns.

Figure 1. Position of Risk-Based Portfolios when Asset Returns are Proportional to Risk

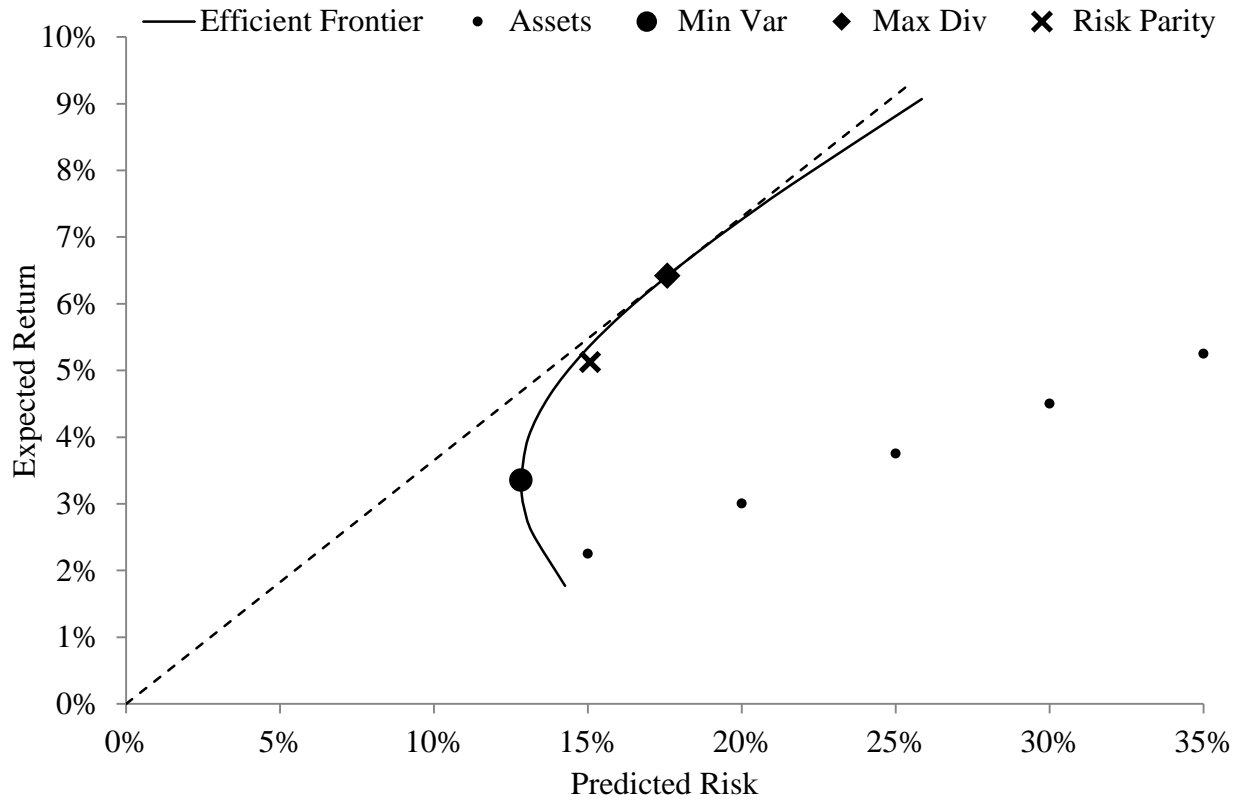
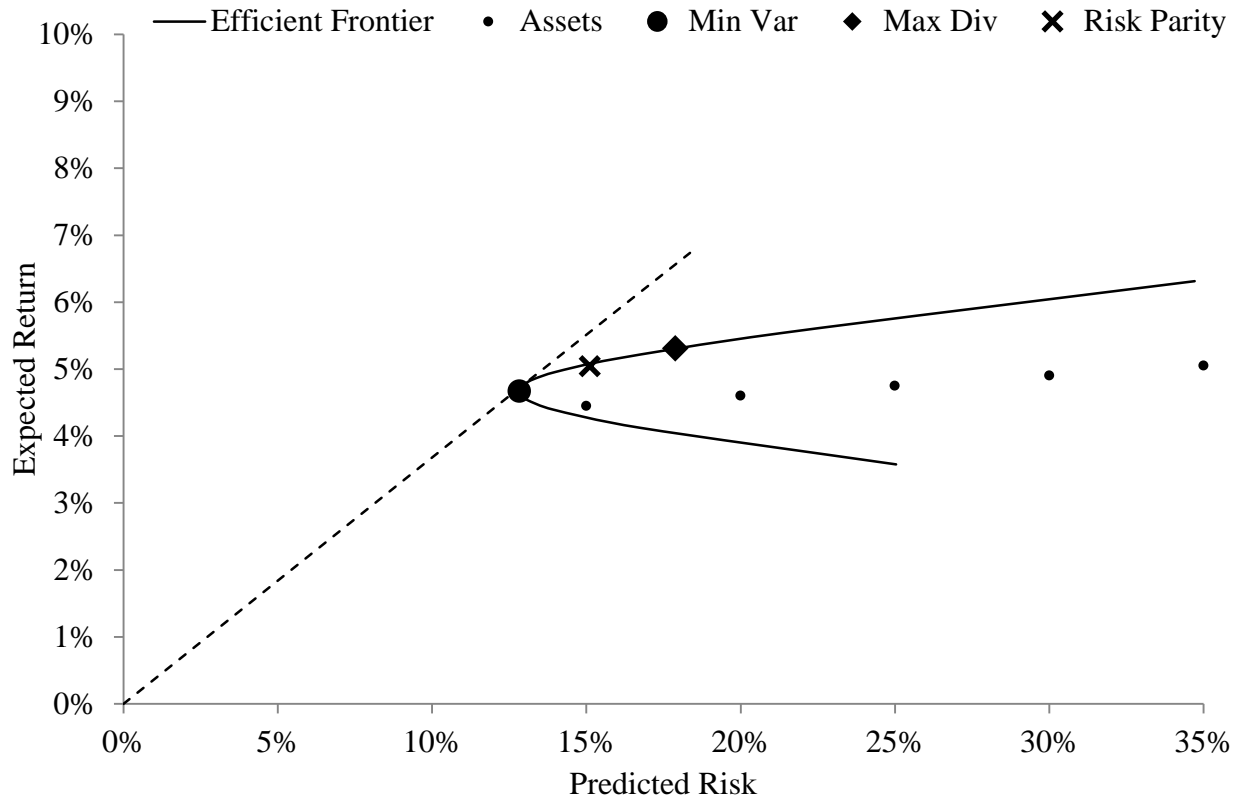


Figure 1 is drawn under the assumption that expected asset returns are perfectly proportional to total asset risk, one of the motives behind the design of the Maximum Diversification objective function. While somewhat similar to the equilibrium result of the Capital Asset Pricing Model, the well-known CAPM predicts that expected asset returns are linear in exposure to systematic risk or beta, not *total* asset risk. Because of the assumption about expected returns in Figure 1, the Maximum Diversification portfolio is on the efficient frontier at the point of tangency or maximum Sharpe ratio. As always, the Minimum Variance portfolio in Figure 1 is at the left-most tip of the efficient frontier curve. The Risk Parity portfolio in Figure 1 is at the left-most tip of the efficient frontier curve. The Risk Parity portfolio has an expected return and risk that is between the other two portfolios, and notably

does *not* lie on the efficient frontier. Except for the special case of “constant correlation”, Risk Parity does not produce a portfolio that has the lowest possible risk for a given expected return.¹

Figure 2. Position of Risk-Based Portfolios when Asset Returns are Increasing with Risk



In Figure 2, expected asset returns are still assumed to be increasing in asset risk, but are not perfectly proportional to risk. As a result, the efficient frontier curve is flatter than in Figure 1, and the Maximum Diversification portfolio is no longer at the point of tangency or maximum Sharpe ratio. On the other hand, the Minimum Variance portfolio is now much closer to (although not exactly at) the point of tangency due to the flatness of the curve. The Risk Parity portfolio still lies between the two other portfolios, and although not as visually apparent in Figure 2, is still slightly off the efficient frontier curve. If one assumes no ability to forecast relative returns, an Information Coefficient of zero in the Grinold (1994) framework, the

¹ Risk Parity portfolios are mean-variance efficient and equivalent to Maximum Diversification portfolios under the assumption of constant correlation (Choueifaty and Coignard, 2008).

efficient frontier curve collapses to a line. Under this condition (not shown), expected asset returns are all equal and the Minimum Variance portfolio is exactly on the point of tangency.

Figure 3. Position of Risk-Based Portfolios when Asset Returns are Decreasing with Risk

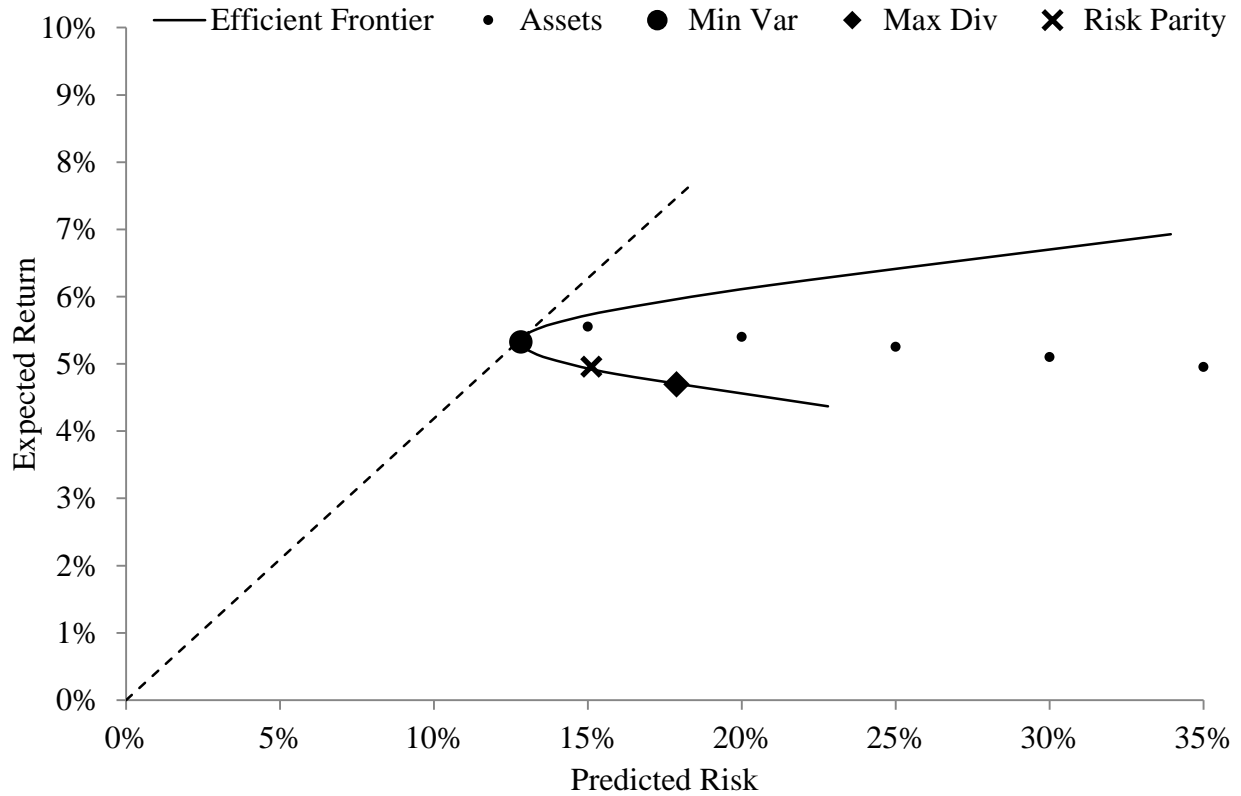


Figure 3 is similar to Figure 2, but with the assumption that expected asset returns are *decreasing* in risk. In this case, the Maximum Diversification portfolio is on the lower side of the efficient frontier curve, often not shown in traditional risk-return charts because a portfolio would not be intentionally constructed to be on that part of the curve. For similar reasons, the Risk Parity portfolio is also on the “wrong side” of the Efficient Frontier, although being slightly off the curve is now helpful. While the situation illustrated in Figure 3 may seem perverse, the pattern is consistent with the low-risk anomaly in equities that we review next.

Figure 4 shows the realized average return and risk of three asset classes using Ibbotson returns from 1968 to 2010, which exhibit the traditional relationship between risk and reward. Starting with T-bills (with an excess return of zero), Figure 4 shows that Treasury bonds, large-capitalization stocks, and small-cap stocks all have similar Sharpe ratios. To provide focus on

this empirical fact, Figure 4 includes a Sharpe ratio line with a slope of 0.30. As is common practice, the realized returns shown in Figure 4 are average annualized (multiplication by 12) monthly returns, although investors may be more interested in *compound* returns if they have a longer holding period. Thus, Figure 4 also includes the annual compound returns associated with each portfolio (position shown with an x), which are lower than arithmetic average returns.² As can be seen in Figure 4, the association between risk and return is even more consistent (i.e., linear) using compound returns.

Figure 4. Asset Class Portfolios from 1968 to 2010

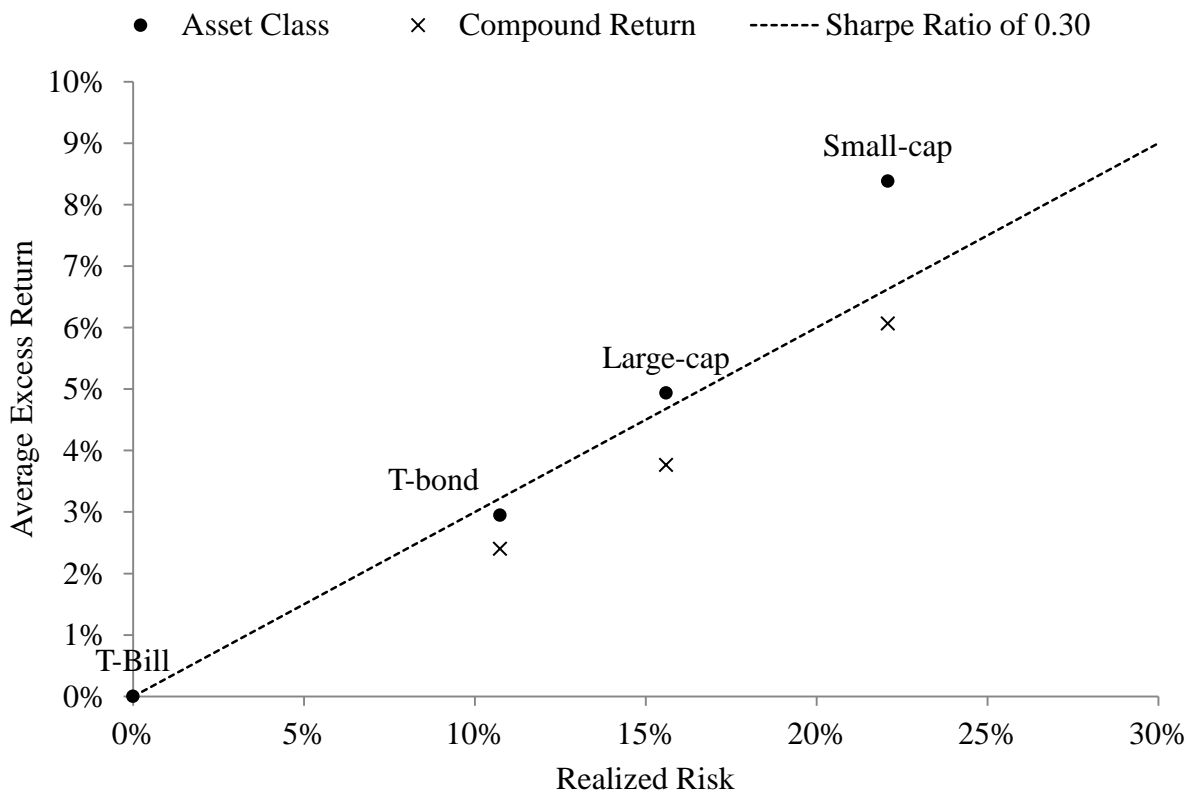
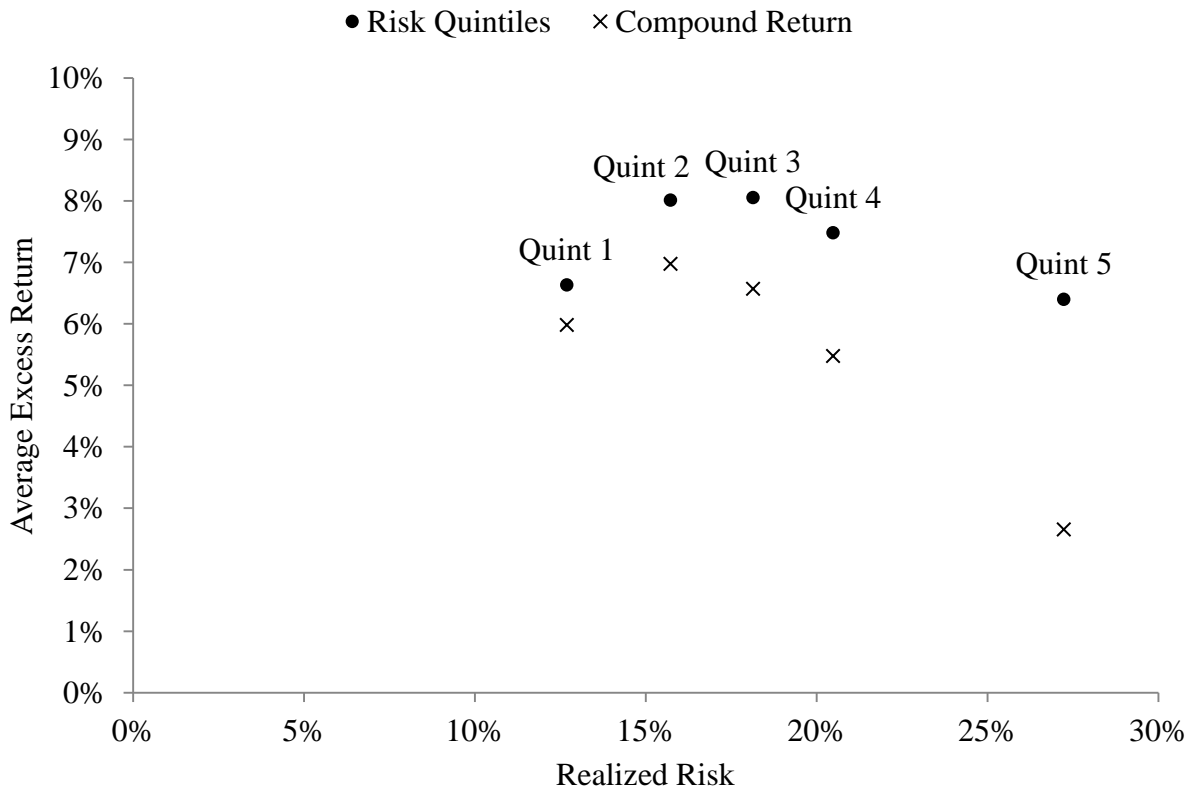


Figure 5 illustrates the “risk anomaly” first documented by Ang, Hodrick, Xing, and Zhang (2006) within the cross-section of stocks. The largest 1000 common stocks in the CRSP database are sorted into quintiles each month based on their historical volatility, specifically the

² We show exact compound returns, but a common rule of thumb, depending on the distribution of returns, is that the compound return is equal to the arithmetic mean return minus half the return variance. The compound returns in Figure 4 are roughly consistent with this rule.

excess return standard deviation over the last 60 months. Figure 5 plots the *realized* average return and risk from 1968 to 2010 to equally-weighted portfolios within each risk quintile (i.e., 200 stocks per quintile). While the lowest and second-to-the-lowest risk portfolios (i.e., Quint 1 and Quint 2) have the traditional pattern, the other three portfolios have an *inverse* relationship between risk and return. The potentially perverse relationship between risk and reward is even worse with compound rather than simple arithmetic realized returns, as shown with an x for each portfolio in Figure 4. The high ex-ante risk quintile does in fact have the highest realized risk at about 27 percent, but with a compound annual excess return of less than 3 percent for the last 43 years. While inconsistent with long-standing academic views of risk and return, Baker and Wurgler (2011) provide a reasonable explanation for the volatility anomaly based on individual investor preferences for high-risk stocks, and constraints associated with benchmarking that prohibit larger institutions from fully exploiting the anomaly. Frazzini and Pedersen (2011) also provide an explanation for the anomaly based on the lack of access to leverage for individual investors, who bid up the price (i.e., lower the subsequent return) of high risk stocks.

Figure 5. Risk Quintile Equity Portfolios: 1968 to 2010



The illustrative charts (Figures 1 to 3) and empirical results (Figures 4 and 5) in this section focused on expected and realized average returns, in contrast to the risk parameters that are the sole inputs to risk-based portfolio construction. The structure of expected returns within the equity market is an ongoing source of debate in academic finance literature, and subject to one's ability to forecast returns in active management practice. In addition, the illustrative charts are based on unconstrained long/short portfolios and small investable sets. We now turn to analytic solutions of large-sample long-only portfolios in order to gain more perspective on their risk properties, including asset weight concentration.

Analytic Solutions to Risk-Based Portfolios

We first review properties of the single-factor risk model and acquaint the reader with our mathematical notation. Under a single-factor asset covariance matrix, individual assets have only one source of common risk, resulting in the familiar decomposition of the i^{th} asset's total risk into systematic and idiosyncratic components

$$\sigma_i^2 = \beta_i^2 \sigma_F^2 + \sigma_{\varepsilon,i}^2. \quad (1)$$

In Equation 1, σ_F is the risk of the common factor, for example a capitalization-weighted market portfolio, and $\sigma_{\varepsilon,i}$ is the i^{th} asset's idiosyncratic risk. The asset's exposure to the systematic risk factor, β_i , is by definition equal to the ratio of asset risk to factor risk, multiplied by the correlation coefficient between the asset and the risk factor,

$$\beta_i = \frac{\sigma_i}{\sigma_F} \rho_i. \quad (2)$$

The single-factor model also yields simple relationships for the pair-wise association between any two assets. Specifically, the covariance between two assets, i and j , is $\sigma_{i,j} = \beta_i \beta_j \sigma_F^2$, and the correlation coefficient is the product of their correlations to the common factor, $\rho_{i,j} = \rho_i \rho_j$.

As shown in Clarke, de Silva, and Thorley (2011), and reviewed in the technical appendix, individual asset weights in the long-only Minimum Variance portfolio under a single factor risk model are

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2} \left(1 - \frac{\beta_i}{\beta_L} \right) \text{ for } \beta_i < \beta_L \text{ else } = 0 \quad (3)$$

where β_L is a “long-only threshold” beta and σ_{MV} is the risk of the minimum variance portfolio. According to Equation 3, individual assets are only included in the long-only portfolio if their factor beta is lower than the threshold beta, β_L , which for large investment sets can exclude a majority of the assets. High idiosyncratic risk in the denominator of the first term of Equation 3 lowers the asset weight, but by itself cannot drive an asset out of solution. The second term, in the parentheses of Equation 3, indicates that asset weights increase from zero with the only other asset-specific (i.e., subscripted by i) parameter, β_i , and that the highest weight is given to the lowest beta asset. In fact, in the absence of the cross-sectional variations due to idiosyncratic risk, Minimum Variance portfolio asset weights fall on a kinked line when plotted against factor beta, visually like the expiration-date payoff profile to a put option. Specifically, weights are zero above the long-only threshold beta, β_L , and lie on a line with a slope of $-1/\beta_L$ for betas below the long-only threshold, including negative beta assets if any.

The technical appendix shows that asset weights in the Maximum Diversification portfolio are similar in form to the Minimum Variance weights, but key off of the *correlation* of the asset to the common risk factor

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2} \frac{\sigma_i}{\sigma_A} \left(1 - \frac{\rho_i}{\rho_L} \right) \text{ for } \rho_i < \rho_L \text{ else } = 0 \quad (4)$$

where ρ_L is a “long-only threshold” correlation and σ_{MD} is the Maximum Diversification portfolio risk. According to Equation 4, individual assets are only included in the long-only Maximum Diversification portfolio if their correlation to the common risk factor is lower than the long-only threshold correlation. As in Equation 3 for Minimum Variance portfolio weights, high idiosyncratic risk in the denominator of the first term of Equation 4 lowers the asset weight, but does not by itself drive an asset out of solution.

Equation 4 also has a middle term; the ratio of the asset risk, σ_i , to the weighted average risk of the assets in the long-only portfolio, σ_A . This middle term, like the first term, sizes the weight rather than dictating the asset's inclusion in the Maximum Diversification portfolio. However, total asset risk in the numerator of the middle term does tend to offset idiosyncratic risk, so that in general Maximum Diversification portfolio weights are proportional to the inverse of asset return standard deviation, as opposed to the inverse of variance in Minimum Variance portfolio weights. The result of this structure is that weights in the Minimum Variance portfolio tend to be more concentrated than weights in the Maximum Diversification portfolio, because inverse variance has more extreme values than inverse standard deviation. Finally, the last term in the parentheses of Equation 4 indicates that asset weights increase from zero with lower correlation, and that the highest weight is given to the lowest correlation asset. In the absence of the variations due to idiosyncratic risk, Maximum Diversification portfolio weights lie a kinked line when plotted against correlation rather than beta, including negative correlation assets if any.

The technical appendix provides a new and conceptually important solution for individual assets weights in a Risk Parity portfolio. As explained in the technical appendix, the equation for Risk Parity asset weights is somewhat different in algebraic form than the two objective-function based portfolios;

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2} \left[\left(\frac{\beta_i^2}{\gamma^2} + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2} \right)^{1/2} - \frac{\beta_i}{\gamma} \right] \quad (5)$$

where γ is a constant across all assets, σ_{RP} is the Risk Parity portfolio risk, and N is the number of assets. High idiosyncratic risk in the denominator of the first term of Equation 5 lowers the asset weight, but the similarity to Equations 3 and 4 ends there. As shown using partial derivatives in the technical appendix, asset weights decline monotonically with asset beta, but *all* assets have some positive weight so that the Risk parity portfolio is long-only by definition. For positive beta assets, the square-brackets term of Equation 5 must be positive because the square-root of squared beta over gamma plus some small (divided by N) positive (other variables squared) value is greater than the beta over gamma subtracted at the end. On the other hand, if the asset beta is some small negative number, then the asset weight is clearly positive and large.

In the absence of variations in idiosyncratic risk in Equation 5, Risk Parity weights plotted against factor beta lie along a hyperbolic curve, somewhat like the pre-expiration payoff profile for a put option. In other words, Risk Parity portfolio weights tend asymptotically towards zero with higher beta, and tend asymptotically towards a line with a negative slope for lower beta.

Equations 3, 4, and 5 for individual asset weights provide intuition for the properties of risk-based portfolio construction, as well as formulas for determining weights without the need for an optimizer or complex numerical search routines. In Equations 3 and 4, assets need only be sorted by increasing beta and correlation, respectively, and then compared to the threshold parameters to determine which assets are in solution. The sum of raw weights can then be scaled to one to accommodate the portfolio constants outside the parentheses. Equation 5 for Risk Parity portfolio is inherently different in that portfolio constants, which depend on the final assets weights, are embedding *within* the parentheses, making the weights endogenously determined. However, a fast convergence routine can be designed based on Equation 5 by initially assuming equal weights and then iteratively calculating the portfolio parameters and assets weights until the weights converge to the inverse of N times portfolio beta. Unlike other numerical routines discussed for Risk Parity weights, the calculation of asset weights in Equation 5 is almost instantaneous and can be applied to large (i.e., 1000 asset) investment sets using a simple Excel spreadsheet. While we focus on the intuition of single-factor solutions in the body of the paper, the technical appendix provides a general K -factor solution to the Risk Parity portfolio, as well as the forms for K -factor solutions to the Minimum Variance and Maximum Diversification portfolios.

Ten Asset Hypothetical Example

To illustrate the portfolio weights specified in Equations 3, 4, and 5, we first provide a numerical example for ten hypothetical assets in Table 1, identified with the letters A through J. For example, Asset A has a risk-factor beta of 1.00 and idiosyncratic risk of 40.0%. Using the common factor risk of 10.0%, the total risk of Asset A is 41.2% according to Equation 1, and the correlation to the risk factor is 0.243 according to Equation 2. The other nine assets, B through J, illustrate three values of the common factor beta and three values of idiosyncratic risk in order to contrast and compare the impact on optimal weights for each of the three portfolios.

Table 1. Risk-Based Portfolio Construction for Ten Assets

Asset	Beta	Idio. Risk	Total Risk	Factor Correl.	Portfolio Weights		
					Min Var	Max Div	Risk Par
A	1.000	40.0%	41.2%	0.243	5.9%	9.6%	8.6%
B	0.200	20.0%	20.1%	0.100	34.8%	23.9%	20.1%
C	0.200	30.0%	30.1%	0.067	15.5%	16.7%	13.9%
D	0.200	50.0%	50.0%	0.040	5.6%	10.4%	8.6%
E	1.000	20.0%	22.4%	0.447	23.8%	12.7%	13.3%
F	1.000	30.0%	31.6%	0.316	10.6%	11.3%	10.5%
G	1.000	50.0%	51.0%	0.196	3.8%	8.3%	7.2%
H	2.800	20.0%	34.4%	0.814	0.0%	0.0%	6.7%
I	2.800	30.0%	41.0%	0.682	0.0%	2.6%	6.1%
J	2.800	50.0%	57.3%	0.489	<u>0.0%</u>	<u>4.5%</u>	<u>5.0%</u>
Factor risk		10.0%			100.0%	100.0%	100.0%
Threshold beta		2.721		Port. Risk	12.3%	13.5%	14.2%
Threshold correl.		0.762		Div. Ratio	2.24	2.46	2.38
Gamma		4.108		Effective N	4.5	7.0	8.4

The Minimum Variance (Min Var) weights in Table 1 decrease with idiosyncratic risk for each value of beta, except for the last three assets which are not in solution. The weight for each asset with a beta of 1.000 is lower than the weight of each matching idiosyncratic risk asset with a beta of 0.200. The Minimum Variance weight for the three assets with high betas is zero because 2.800 exceeds the long-only threshold value of 2.721, calculated from the individual asset risk parameters as specified in the technical appendix. The Maximum Diversification (Max Div) weights also decrease with higher idiosyncratic risk, and decrease with higher correlation, but only Asset H has a correlation coefficient that exceeds the long-only threshold correlation of 0.762, calculated from the individual asset risk parameters as specified in the technical appendix.

Given that the risk factor correlations in Table 1 are a function of both systematic (i.e., beta) risk and idiosyncratic risk, the asset order sorted by correlation is different than the order sorted by beta. The implication is that an asset may be in solution for the Maximum Diversification portfolio, but not in the Minimum Variance portfolio, and vice-versa. The within-set variation in asset weights in Table 1 is more extreme in the long-only Min Var

portfolio compared to the Max Div portfolio, consistent with weighting loosely associated with inverse variance as opposed to inverse standard deviation. Assets in the Risk Parity portfolio are all positive, and follow the same pattern as the other two weight sets, with high asset risk of either type leading to a lower asset weight. However, the weights in the Risk Parity portfolio are not aligned with either beta or correlation. Rather, they are proportional to the inverse of each asset's beta with respect to the final Risk Parity portfolio. Table 1 also provides the Risk Parity constant (γ) value 4.108 shown in Equation 5, based on calculations specified in the technical appendix.

The final panel of Table 1 shows the predicted portfolio risk, Diversification Ratio, and Effective N, for each of the three risk-based portfolios. The 12.3 percent risk of the Minimum Variance portfolio is by design the lowest of the three portfolios, while the 2.46 Diversification Ratio for the Maximum Diversification portfolio is the highest based on its objective function. The risk of the Risk Parity portfolio is higher than the other two, while the Diversification Ratio lies between them, although that order is specific to this numerical example. As explained in the technical appendix, Effective N is a measure of portfolio concentration, with lower values indicating higher concentration of weights. The 4.5 Effective N of the Minimum Variance portfolio is the lowest of the three because the portfolio excludes more assets, and has more extreme variation of positive weights. On the other hand, the 8.4 Effective N of the Risk Parity portfolio is the highest of the three, a pattern that is typical, although not universal. Specifically, the Risk Parity portfolio has all assets in solution by definition, but Effective N also depends on the magnitude of asset weights, and large asset sets will have a many Risk Parity weights close to zero.

One Thousand Asset Empirical Example

While ten assets may be indicative of an asset class allocation problem, security selection typically involves a much bigger investable set. In this section, we use Equations 3, 4, and 5 to construct portfolios from the largest 1000 U.S. common stocks in the CRSP database at the end of each month from 1968 to 2010. To keep the risk model estimation process as simple as possible, beta and idiosyncratic risk are based on standard OLS regressions for the prior 60 months, using the value-weighted CRSP index as the common factor. To translate historical OLS betas into predicted betas, the entire set of 1000 is adjusted $\frac{1}{2}$ towards 1.000 each month.

Similarly, the set of log idiosyncratic risks is adjusted $\frac{1}{3}$ towards the average log idiosyncratic risk for that period.³ The results in this paper are generally robust to the specific shrinkage process, although some kind of Bayesian adjustment is needed to avoid extreme predicted risk parameter values and thus extreme weights. A full 60 months of prior return data is required for a stock to be considered for the portfolios, but no other data scrubbing, stock selection, or maximum weight constraints are employed.

Figure 6. Predicted Asset Risks from 1968 to 2010

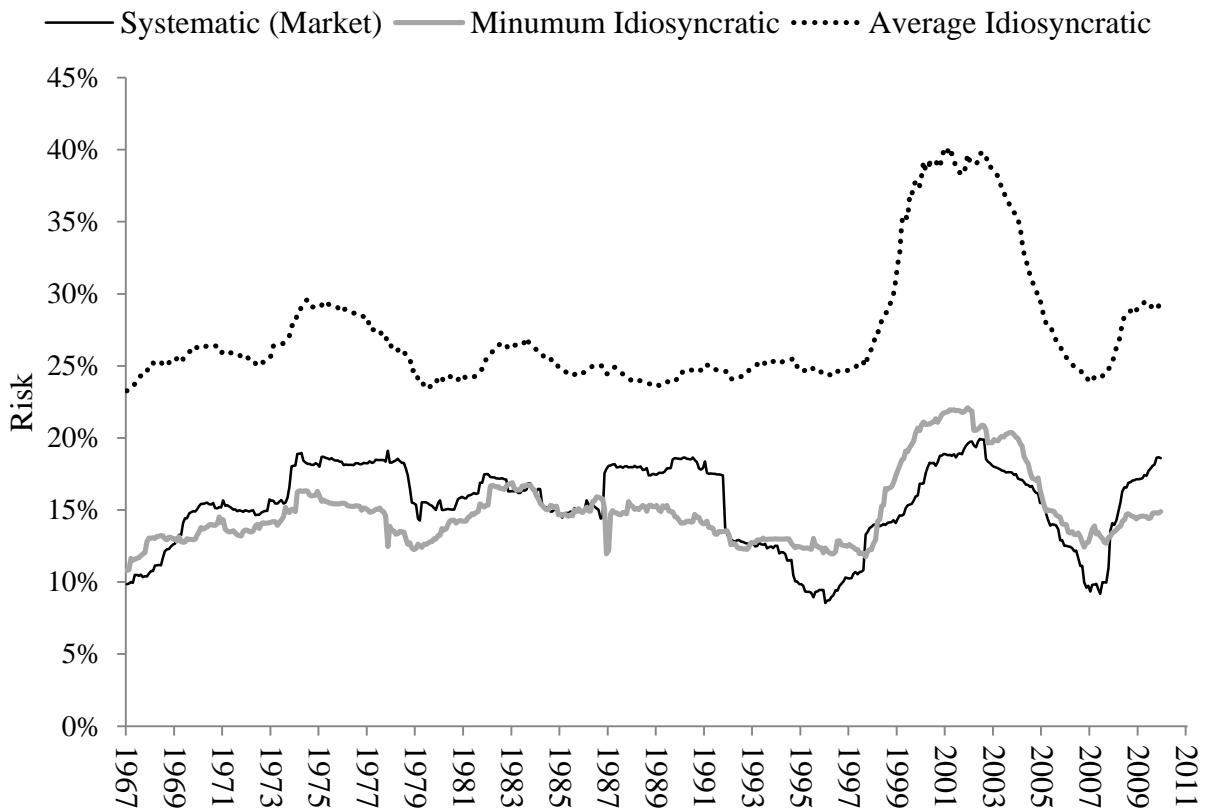


Figure 6 shows predicted systematic risk, minimum idiosyncratic risk, and the average idiosyncratic risk, for our single factor model. Systematic (i.e., market) risk, based on the realized risk over the prior 60 months, varies between about 10 and 20 percent. The minimum

³ The “shrinkage” of historical betas for purposes of risk prediction is similar to the Bloomberg rule of adjusting historical beta values $\frac{1}{3}$ towards one. We use $\frac{1}{2}$ instead of $\frac{1}{3}$ based on observed values of the coefficient in cross-sectional regressions of 60-month realized betas on 60-month historical betas for 1000 stocks. The choice of $\frac{1}{3}$ shrinkage towards the mean for log idiosyncratic risk is also based on historical regression values. We shrink using logs because idiosyncratic risk is by definition a positively valued variable and thus highly skewed.

idiosyncratic risk (out of all 1000 stocks) also varies in the same general range and follows a similar pattern over time. The average idiosyncratic risk runs between 25 and 30 percent, with the notable exception of the turn of the century, when average idiosyncratic risk increases to about 40 percent.

Figure 7. Predicted Asset Betas from 1968 to 2010

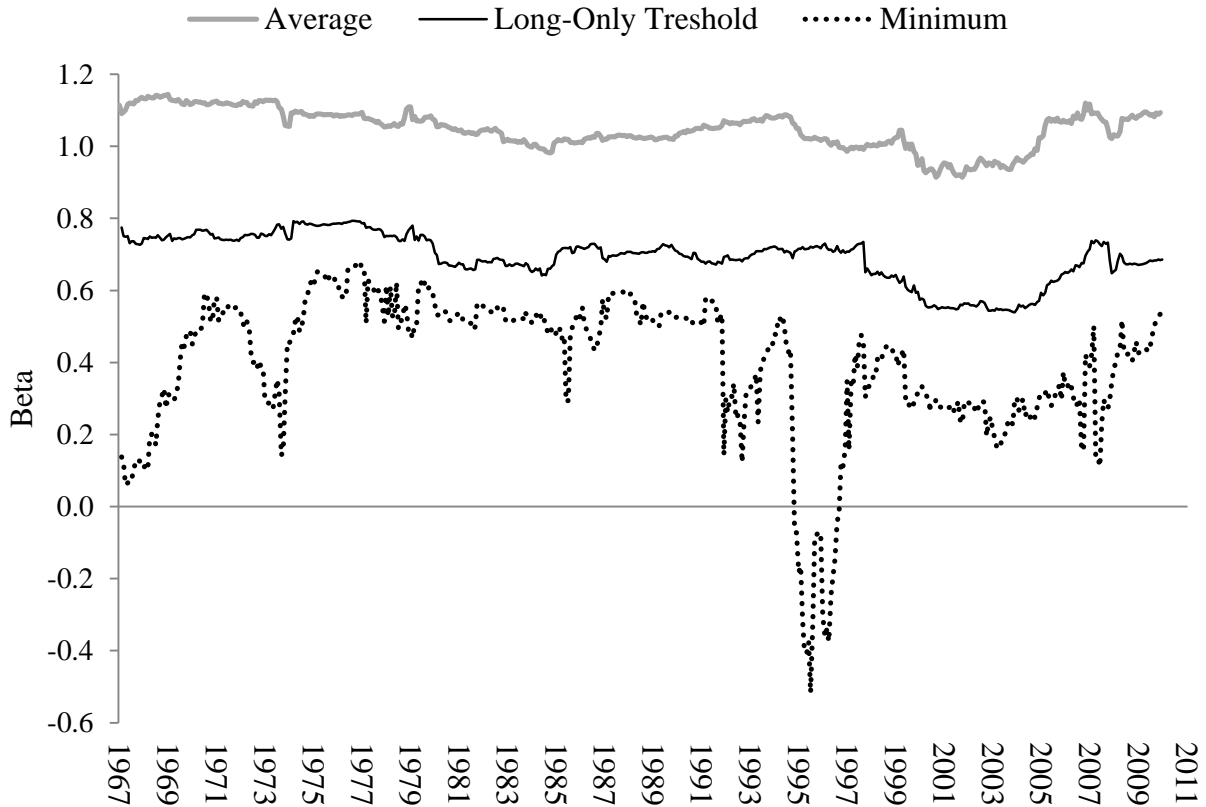


Figure 7 plots the average, long-only threshold, and minimum predicted betas over time for our risk model. The average beta varies around one each month, but is not exactly one as it would be for a value-weighted average. The long-only threshold beta, which dictates which stocks come into the Minimum Variance solution according to Equation 3, ranges between 0.8 and 0.6. The lowest beta out of all 1000 stocks exhibits a lot of variation, ranging between from a little over zero to as high as 0.6, with the notable exception of a period in the mid-1990s when some negative beta stocks are included in the largest 1000 U.S. equities. Further examination indicates that there are just two negative beta stocks, the Best Buy Company and 7 Eleven Incorporated, both counter-cyclical retailers. Other low but not negative beta stocks include

precious metal mining and retail bank holding companies. According to Equation 4, these stocks will receive high weights in the Minimum Variance portfolio. A plot (not shown) of average, long-only, and minimum factor *correlations*, which are important in the construction of the Maximum Diversification portfolio, looks similar to Figure 7.

Figure 8. Effective N for Risk-Based Portfolios from 1968 to 2010

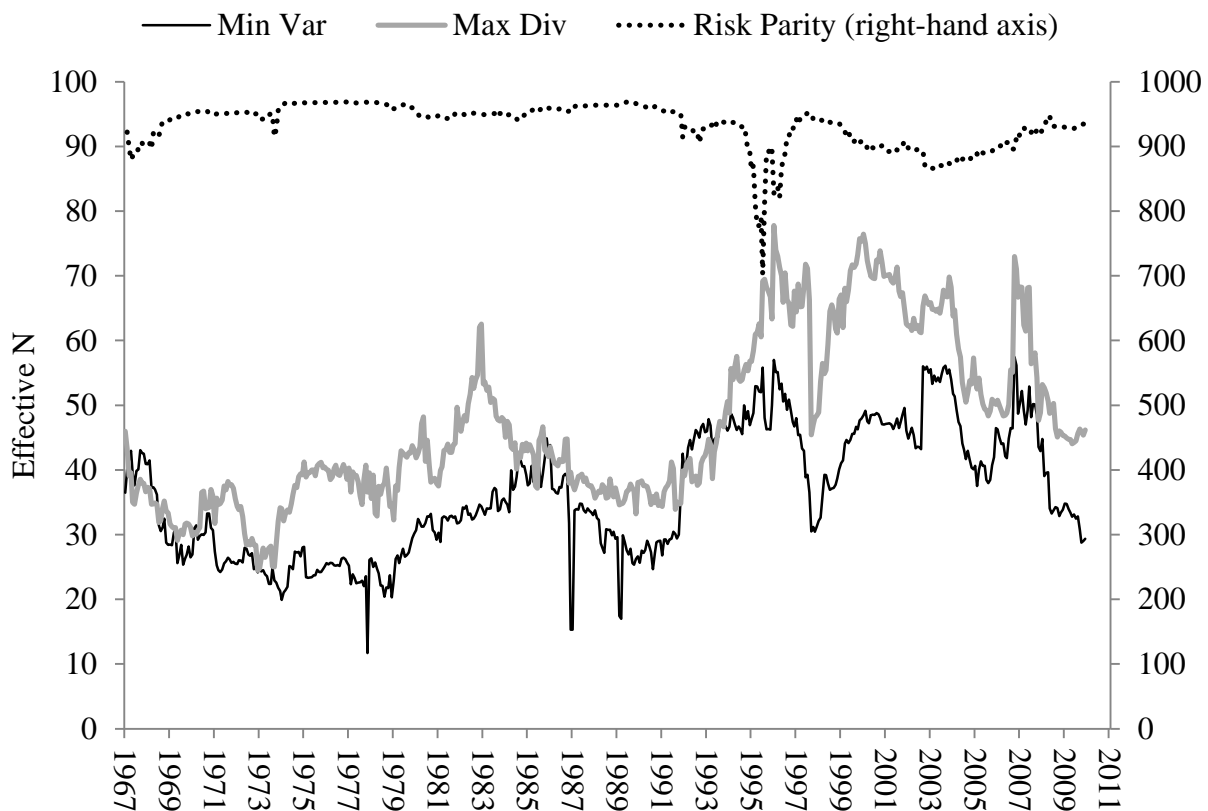


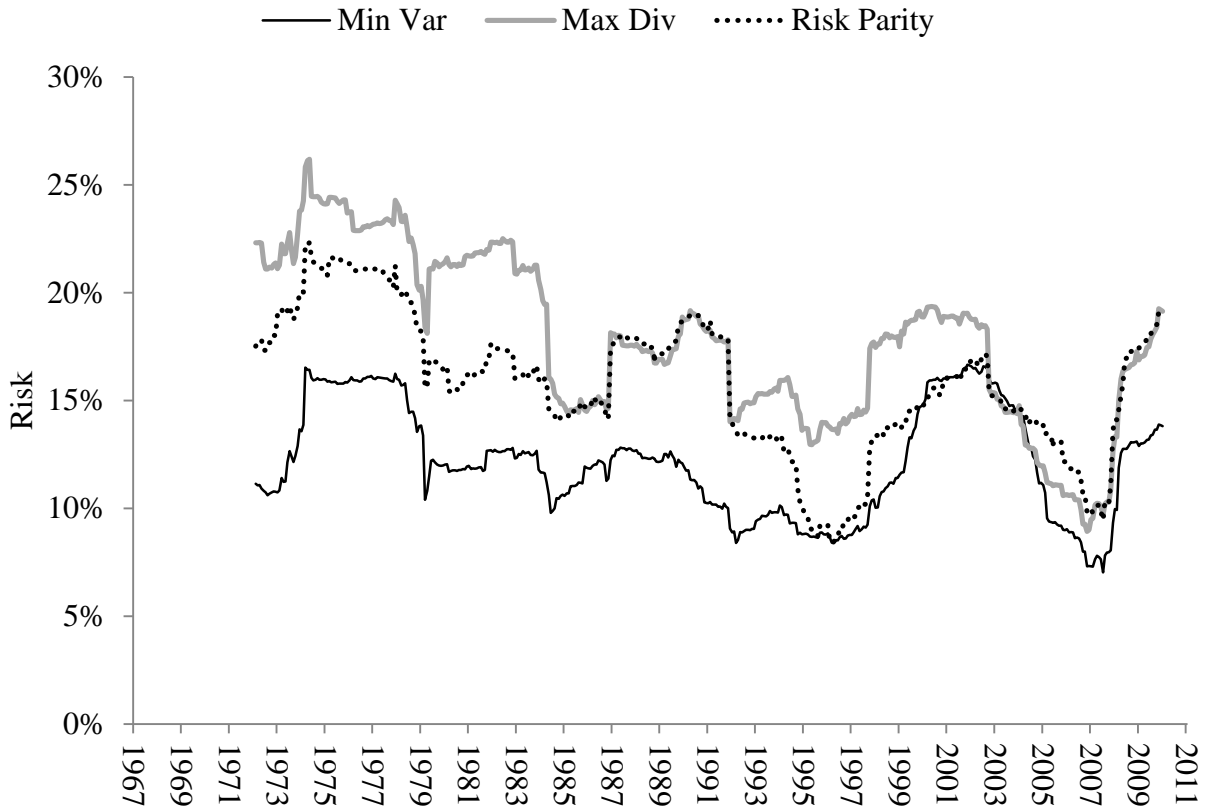
Figure 8 plots the Effective N over time for the three risk-based portfolios constructed using Equations 3, 4, and 5, similar to a chart in Linzmeier (2011). To more closely examine portfolio concentration, the Risk Parity portfolio is scaled on the right vertical axis, separately from the Min Var and Max Div portfolios on the left, because the Effective N values for the Risk Parity are much higher. The Min Var portfolio tends to be more concentrated than the Max Div portfolio, although this is not always the case as seen for short periods in 1968 and again in 1993. The negative betas of the mid-1990's shown in Figure 7 significantly lower the Effective N (increase concentration) of the Risk Parity portfolio, but are associated with an *increase* in the Effective N of the other two portfolios. Figure 6 shows that the mid-1990s was also a period of

low systematic risk, which increases the threshold parameters and thus the number of stocks in solution for the long-only constrained portfolios.

The number of stocks in solution from 1968 to 2010 averaged 63 for the Min Var portfolio compared to 83 for the Max Div portfolio, or respectively about 6 percent and 8 percent of the 1000 investable assets. These percentages are in stark contrast to the 70 and 90 percent of assets in solution for the ten-asset example in Table 1. To confirm this pattern, we also examined an investable set of the 100 (rather than 1000) largest U.S. stocks. The number of stocks in solution averaged 25 for the Min Var portfolio and 43 for the Max Div portfolio, or 25 percent and 43 percent, respectively. As explained in the technical appendix, the relative impact of systematic risk increases (impact of idiosyncratic risk decreases) with larger investments sets, lowering the long-only thresholds and decreasing the number of assets that come into solution as a percentage of investable assets.

The prior charts (Figures 6, 7, and 8) illustrated *ex-ante* properties of the three risk-based portfolios. Figure 9 shows the realized risk of all three portfolios over time using a 60-month rolling window, first measured in 1973 (five years after our start date of 1968). With a brief exception in 2003, the risk of the Max Div portfolio is consistently higher than the Min Var portfolio, and usually higher than the Risk Parity portfolio. The realized portfolio risk at each point of time in Figure 9 is by design closely related to the risk that is *predicted* for that portfolio going forward. A chart of these predicted risks indicates that risk is generally underestimated for the Minimum Variance and Maximum Diversification portfolios, but not for the Risk Parity portfolio. Specifically, the average ratio of realized to predicted risk is 1.25 for Minimum Variance, 1.54 for Maximum Diversification, and 1.03 for Risk Parity. The underestimation of risk for the two objective function based portfolios is a natural consequence of using the same risk model for estimation as well as optimization. Less clear is why the understatement of the Maximum Diversification risk is more pronounced, although it may be a consequence of having risk estimates in the numerator and as well as the denominator of the objective function.

Figure 9. Realized Portfolio Risk from 1973 to 2010



The average performance from 1968 to 2010 of the three risk-based portfolios, as well as the market (value-weighted) portfolio are given in Table 2, and shown in Figure 10. Based on DeMiguel, Garlappi, and Uppal (2009) we also include an Equal-Weighted portfolio as a form of “naive diversification”. While high average returns are not the explicit goal of risk-based portfolio construction, all three risk-based portfolios outperform the excess market return of 5.2 percent. The Minimum Variance and Maximum Diversification portfolios both have average returns of 5.6 percent, while the Risk Parity portfolio has a return of 7.3 percent, closely matching the return on an Equally-Weighted portfolio. However, the objective of low risk is best achieved by the Minimum Variance portfolio, with a realized risk of 12.6 percent compared to the market risk of 15.6 percent. Alternatively, as reported in Table 2, the Minimum Variance and Risk Parity portfolios are essentially tied on a Sharpe ratio basis. Note that axes in Figure 10 start at a 3 percent average return and 10 percent risk, and end at 8 and 24 percent respectively, so a diagonal line would represent a Sharpe ratio of 0.30.

Table 2. Performance of 1000 Stock Portfolios from 1968 to 2010

	Market (Value-Weighted)	Equal Weighted	Minimum Variance	Maximum Diversification	Risk Parity
Average Return	5.2%	7.3%	5.6%	5.6%	7.3%
Standard Deviation	15.6%	17.9%	12.6%	19.0%	16.7%
Sharpe Ratio	0.33	0.41	0.44	0.30	0.44
Beta	1.00	1.08	0.52	0.93	1.01
Compound Return	4.0%	5.8%	4.9%	3.8%	6.1%

Figure 10. Risk-Based Equity Portfolio Performance from 1968 to 2010

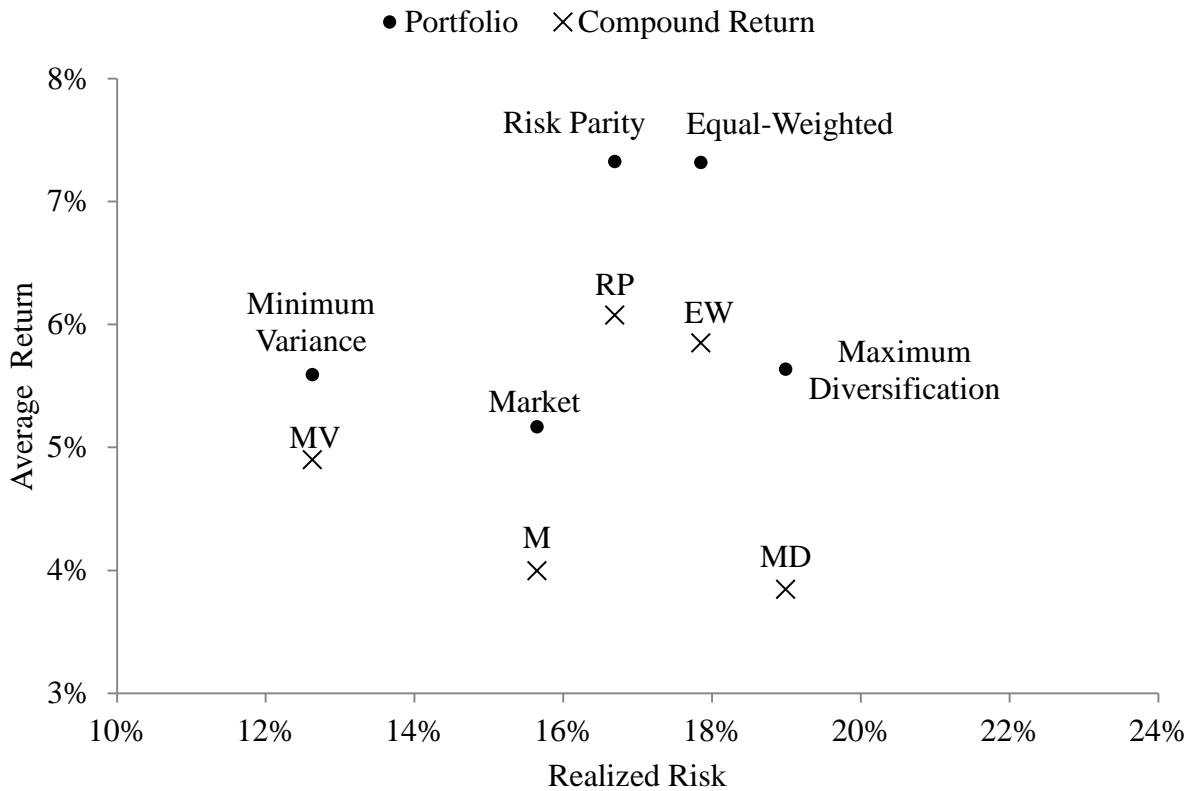
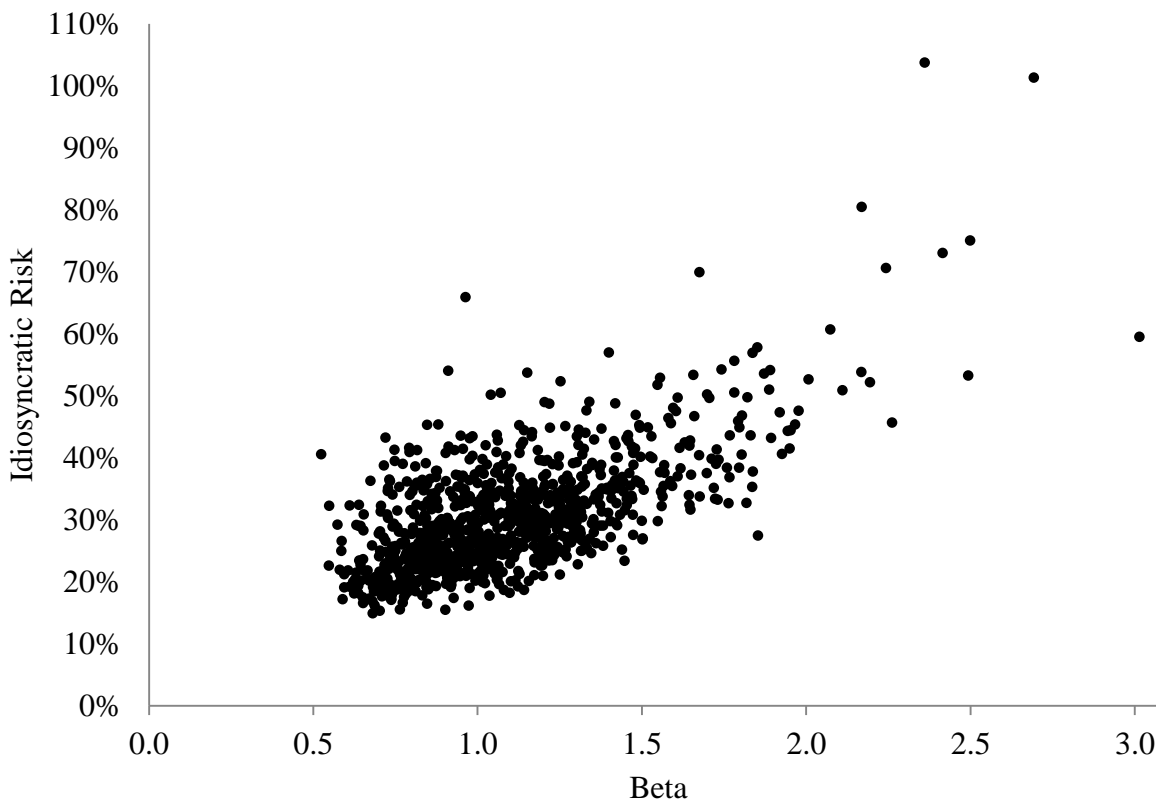


Table 2 includes two other portfolio performance statistics; realized market beta and compound average return. As in Figures 4 and 5, the compound return of each portfolio is also shown with an x, as well as acronym in Figure 10. The compound return of each portfolio is

lower than the average return as a function of portfolio risk, which increases the Sharpe ratio of the Minimum Variance (MV) relative to Risk Parity (RP) portfolio, and shows a notably low relative Sharpe ratio for the Maximum Diversification (MD) portfolio. Consistent with Choueifaty and Coingnard (2008), we do find in sensitivity analysis that the Maximum Diversification portfolio fares better using the top 500 (i.e., S&P 500) rather than the top 1000 (e.g., Russell 1000), although still not as well as the other two portfolios, similar to the findings of Linzmeier (2011). We also note that correlations, which are critical to Maximum Diversification portfolio structure, may be better estimated with a multi-factor model. The market beta of the various portfolios is about one, but with a remarkably lower beta of 0.52 for the Minimum Variance portfolio, a key source of its low realized risk, as explained in Clarke, de Silva, and Thorley (2011).

Figure 11. Beta and Idiosyncratic Risk for 1000 U.S. Stocks in 2010



Weighs as a Function of Risk

We now turn to an examination of weight structure of the three risk-based portfolios on two specific dates, the last month (December 2010) in the sample, and a month with negative

beta observations (December 1995). To support the subsequent weight charts, Figure 11 plots the predicted beta and idiosyncratic risk of all 1000 stocks in December 2010. Beta ranges from about 0.5 to 3.0, and idiosyncratic risk ranges from about 15 percent to over 105 percent. The ranges for December 2010 are a bit high by historical standards, although not as high as the around the turn of the century. As is typical over time, the scatter-plot in Figure 11 shows that the two measures of asset risk are somewhat correlated and that beta, as well as idiosyncratic risk, is positively skewed. The predicted common factor risk at the end of 2010 is 18.6 percent, relatively high by historical standards. As we shall see, the impact of the higher systematic risk in 2010 leads to even more concentration in the Minimum Variance and Maximum Diversification portfolio than is typical over time.

Figure 12. Risk-Based Portfolio Weights and Beta in 2010

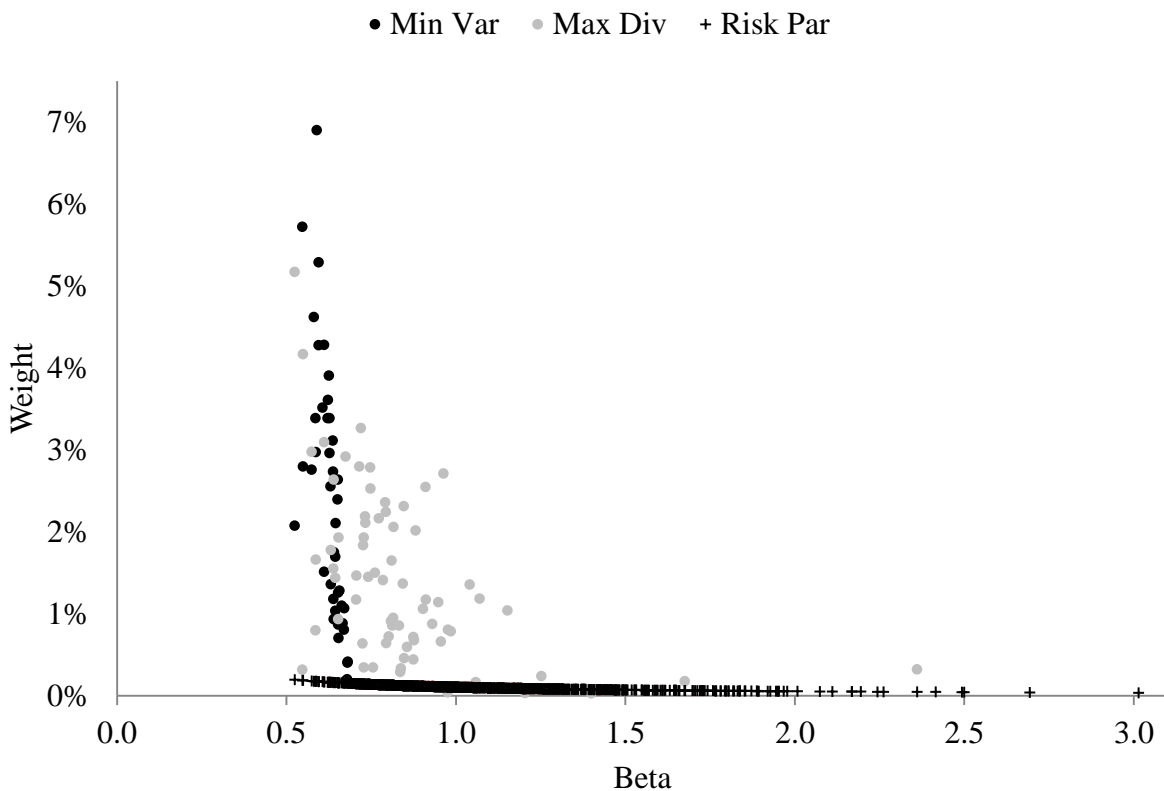
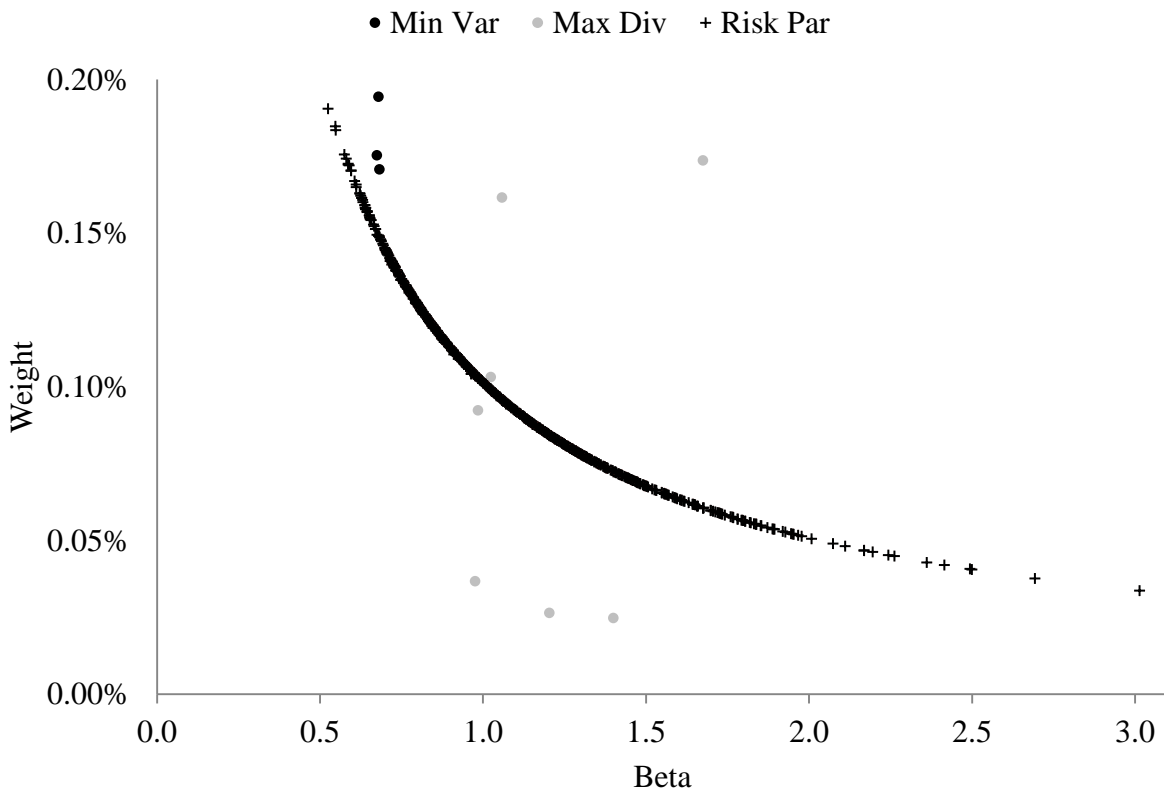


Figure 12 shows asset weights plotted against beta for the Min Var portfolio (black dots), the Max Div portfolio (grey dots), and the Risk Parity portfolio (+ signs). Only 43 of the 1000 stocks are in solution for the long-only Min Var portfolio in December 2010, ranging from a weight of about 7 percent to zero. The stocks with positive Min Var weights all have betas

below of the long-only threshold beta of about 0.7 for this period, in accordance with Equation 3. The Min Var weights tend to fall on a negatively sloped line, also in accordance with Equation 3, although the correspondence is not perfect because of the impact of idiosyncratic risk. The combination of a few stocks in solution and some relatively large weights leads to an Effective N for the Min Var portfolio of just 29 in December 2010. The Max Div portfolio in Figure 12 has 72 stocks in solution, with weights ranging from a little over 5 percent to zero. The stocks with positive Max Div weights tend to have low betas, but several have high betas, including one stock with a beta of 2.4. When Max Div weights are plotted against factor correlation (not shown) instead of beta, the positive weights are all associated with stocks that have correlations below the long-only threshold value of about 0.42 for this period, in accordance with Equation 4. For example, the 2.4 beta stock mentioned above has the highest idiosyncratic risk in this set (see highest dot in Figure 11) leading to relatively low correlation to the market and a relatively high Max Div weight (see the right-most grey dot in Figure 12).

Figure 13. Risk-Based Portfolio Weights and Beta in 2010 (expanded view)



The Risk Parity portfolio weights in Figure 12 are all positive, but even the largest is below 0.20 percent. As a result, the Effective N of the Risk Parity portfolio is 934, close to an equally-weighted portfolio which by definition has an Effective N of exactly 1000. Figure 13 shows an expanded view of the bottom of Figure 12 to more closely examine the Risk Parity weights. The Risk Parity weights are remarkably aligned with factor beta along a curve that has the functional form indicated by Equation 6. The Risk parity portfolio includes all 1000 assets, with no single asset having an exceptionally large weight, so the impact of the idiosyncratic risk on the weight of any asset in the Risk Parity portfolio is negligible. The lower concentration (higher Effective N) does *not* mean, however, that the Risk Parity portfolio has less ex-ante risk than the other two portfolios. The Minimum Variance portfolio has the lowest ex-ante risk of all three (or any other long-only portfolio) by design, and will generally have the lowest risk ex-post, depending on the accuracy of the risk model. Finally, we note as discussed in Lee (2011) that the Risk Parity portfolio weights are perfectly aligned with the inverse of another asset beta; the beta of the stock to the final portfolio.

Figure 14. Risk-Based Portfolio Weights and Idiosyncratic Risk in 2010

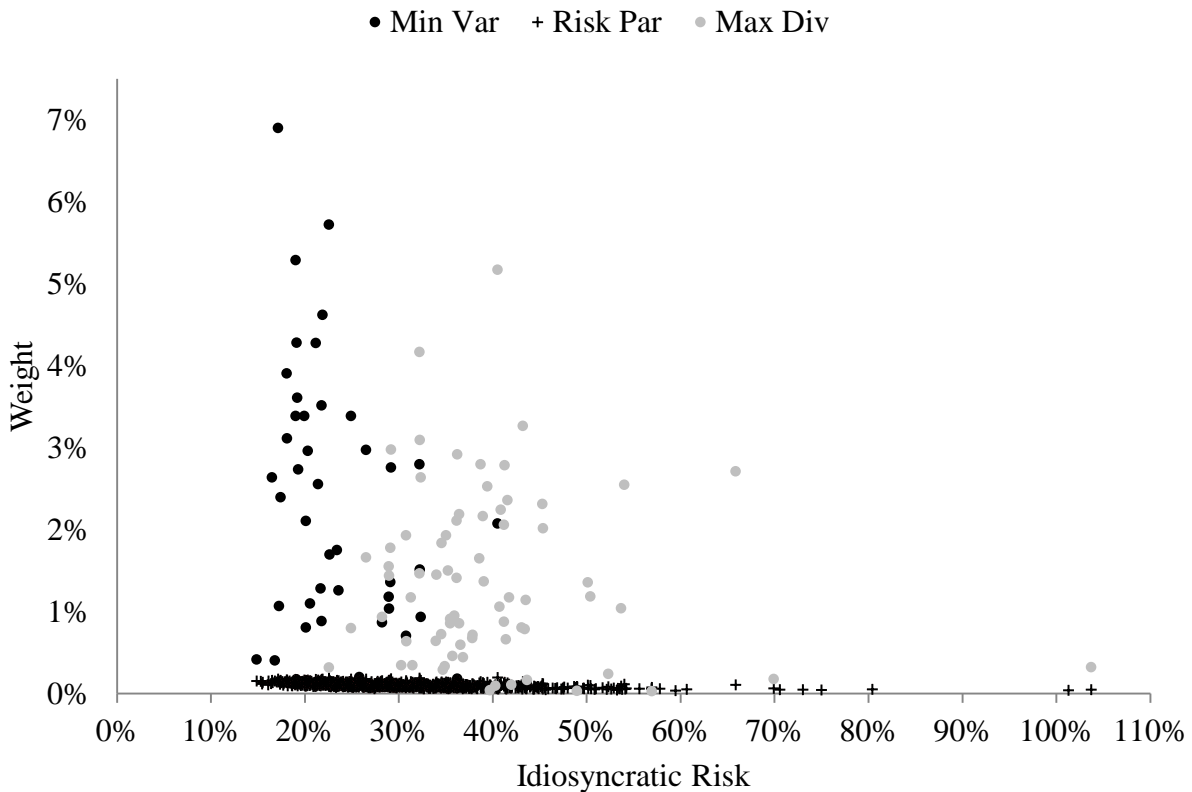


Figure 14 plots the three risk-based portfolios weights in 2010 against *idiosyncratic* risk, similar to the *systematic* risk plot in Figure 12. Weights for the Min Var and Max Div portfolios tend to decline with idiosyncratic risk, in accordance with Equations 3 and 4, although the correspondence is not as tight as with systematic risk. An expanded view of the Risk Parity weights in Figure 14 indicates that they, like Min Var and Max Div weights, also decline with idiosyncratic risk, although the pattern is not tight. Indeed, the pattern of Risk Parity weights in an expanded view looks much like a transposed (axis swapped) version of Figure 11.

Figure 15. Beta and Idiosyncratic Risk for 1000 U.S. Stocks in 1995

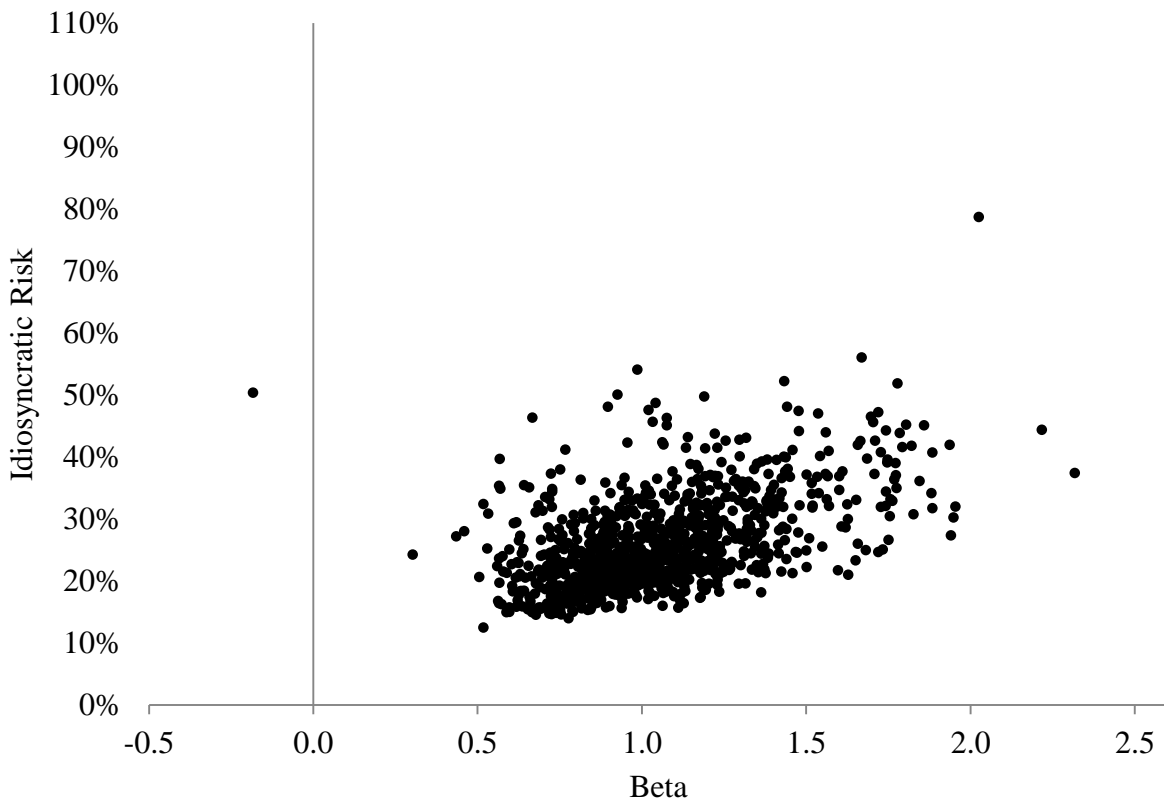


Figure 15 is similar to Figure 11, but for December 1995 rather than December 2010. We chose to examine 1995 based on two interesting aspects of the market in the mid-1990's; relatively low systematic risk of about 10 percent, as shown in Figure 6, and the occurrence of negative beta assets, as shown in Figure 7. Figure 15 shows that idiosyncratic risk is relatively low, with all but one stock under 60 percent. The range of betas is also lower and the horizontal axis is shifted compared to Figure 11 to include the negative beta stock. As in Figure 11, there is some positive cross-sectional correlation between the two (idiosyncratic and systematic)

measures of asset risk, and with the exception of the negative beta asset, the distribution of betas appears to be positively skewed.

Figure 16. Risk-Based Portfolio Weights and Beta in 1995

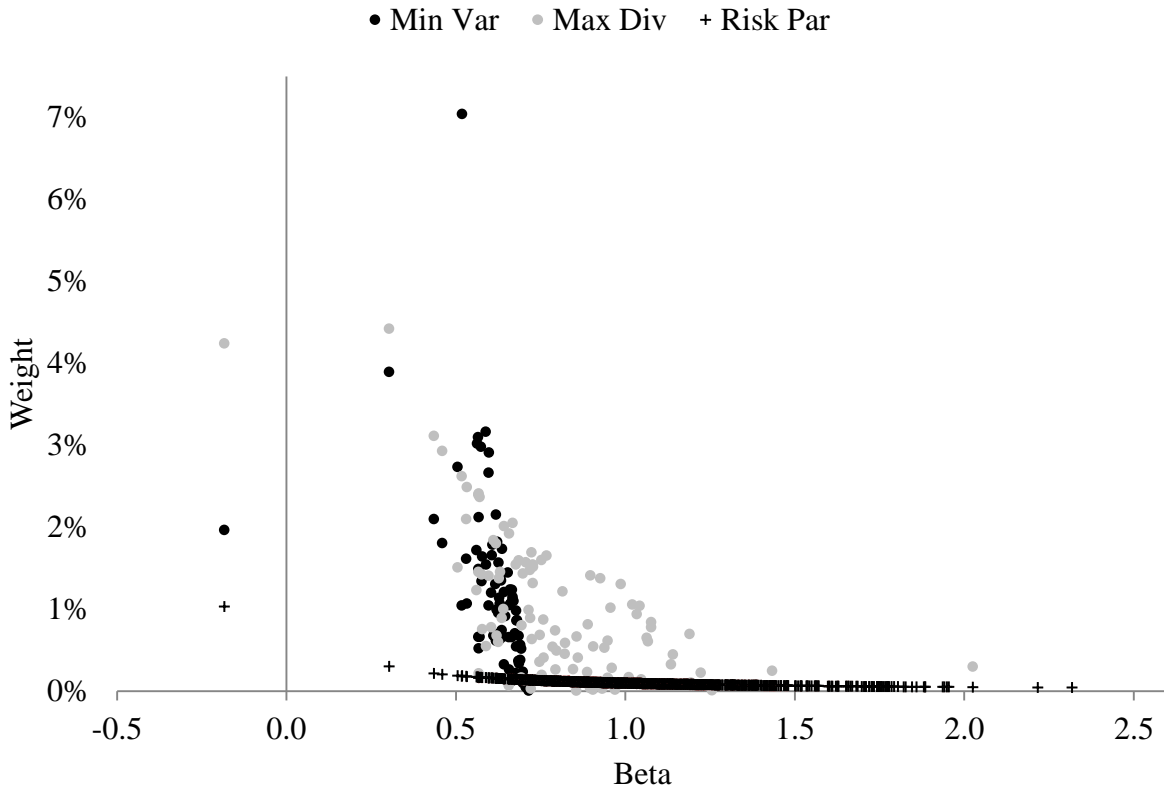


Figure 16 is similar to Figure 12, but plots weights for the three risk-based portfolios against beta in 1995 rather than 2010. As shown in Figure 8, the Effective N of the Minimum Variance and Maximum Diversification portfolios is relative high by historical standards, at 47 and 57 respectively. The actual number of stocks in solution for 1995 (black dots in Figure 16) is 86 for the Minimum Variance portfolio and 100 for the Maximum Diversification portfolio (grey dots in Figure 16). As shown in Figure 8, the Effective N for the Risk Parity portfolio is a relatively low at 864, compared to the more common range of about 900 to 950. Indeed, many of the Risk Parity weights (marked with +) in Figure 16 have material values, with the largest being about 1 percent of the portfolio for the negative beta asset. The weight for this stock (Best Buy Company) is also large in the Minimum Variance and Maximum Diversification portfolios,

but is not the largest weight because of its unusually high idiosyncratic risk of 50 percent, as shown in Figure 15.

Summary and Conclusions

High market volatility, increased investor risk-aversion, and the provocative findings of risk anomalies within the equity market have prompted a surge of empirical research on risk-based portfolio strategies. Simulations of Minimum Variance portfolios over different markets, time periods, and constraint sets have proliferated, with additional interest in Maximum Diversification portfolios and the application of Risk Parity to equity security selection. Most of these empirical studies confirm the finding that Minimum Variance and other risk-based portfolio structures have historically done quite well, due to one or the other linear versions of the “volatility anomaly” or other more dynamic aspects of equity market history. Such studies, however, depend on the specification of the risk model, often proprietary or commercially-based, and individual position limits, in addition to the usual vagaries of empirical work. While some analytic perspectives on the properties of objective-function (i.e., Minimum Variance and Maximum Diversification) portfolios has been published using standard unconstrained portfolio theory, implementation is inevitably in a long-only format leading to portfolios that are materially different from their long/short counterparts. In addition, little analytic work on large-set Risk Parity portfolio construction has been developed, beyond the definitional property that total versus marginal risk contributions define the weight structure.

This paper provides analytic solutions to long-only constrained Minimum Variance and Maximum Diversification portfolios, as well as Risk Parity which is long-only by definition. The optimal weight equations are not strictly closed form, even for the single-factor risk model, but do provide helpful intuition on the construction of risk-based portfolios. Rather than simply supplying historical data to a “black box” constrained optimization routine, the analytic solutions allow quantitative portfolio managers to understand why any given investable asset is included in the risk-based portfolio, as well as the reasons for the magnitude of its weight. In addition, the single-factor solution allows for simple numerical calculations of long-only weights for large investable sets, yielding an empirical simulation that is replicable and unbiased. The application of these equations to the largest 1000 U.S. stocks over several decades confirms the ex-post superiority of Minimum Variance portfolios in terms of minimizing risk, as well as the

unintended benefit of higher realized returns. In addition, the first long-term large-sample empirical results on Risk Parity equity portfolios reported in this study shows promise in terms of a high ex-post Sharpe ratio.

The analytic solutions reveal several important aspects of risk-based portfolio construction. Most are intuitive properties elucidated by the algebraic forms of the optimal risk-based weights, while some are more subtle. First, long-only objective-function-based portfolios exclude a large portion of the investable set, with the proportional exclusion becoming greater with the size of the investable set. Second, all three risk-based portfolios have asset weights that decrease with both systematic and idiosyncratic risk, but systematic (i.e., non-diversifiable) risk is the dominant factor, especially for Risk Parity portfolios which include all investable assets. Third, long-only thresholds on asset risk parameters vary with the level of systematic risk, the dispersion of risk-factor exposure (i.e., asset betas), and the general level of idiosyncratic risk over time. In particular, risk-based portfolios become more concentrated with higher systematic risk, greater dispersion of beta, and lower levels of idiosyncratic risk. Higher portfolio concentration does not, however, equate to higher risk ex-ante, or even ex-post in terms of the empirical track record of the single-factor model. Fourth, negative beta assets have particularly large weights (intuitively due to hedging properties) in both objective-function-based portfolio construction and Risk Parity portfolios, although extreme negative beta assets cannot be accommodated in the standard definition of Risk Parity.

As noted in Scherer (2011), the Ang, Hodrick, Xing, and Zhang (2006) historical risk anomaly in the cross-section of stocks can be partially characterized as asset exposure to systematic risk, idiosyncratic risk, or both. However, the inter-temporal dynamics of the market over time, as well as the “second moment” nature of risk (i.e., return variance) makes it unlikely that the anomaly can be completely reduced to a simple linear factor, like value or momentum. In any event, the primary purpose of this study has been to provide new analytic solutions to risk-based portfolios, not just another empirical back-test for domestic stocks. While we do not promote any particular risk-based portfolio strategy or trading rule (e.g., with transaction cost and turnover analysis) the emergence of new objective functions specifically designed to exploit the risk anomaly may constitute a subtle form of data mining. In that regard, Minimum Variance portfolios, specified from the beginning of modern portfolio theory (i.e., 1960s), may be more

robust. On the other hand, Risk Parity, which was conceptualized in the 1990's with asset allocation in mind, but now applied to large-set security portfolios, may also be less susceptible to an ex-post bias. We look forward to more analytic work on risk-based portfolios, as well as empirical studies on various security markets using the numerical recipes for risk-based portfolio construction provided in this study.

Technical Appendix

The Minimum Variance objective function is minimization of ex-ante (i.e., estimated) portfolio risk,

$$\sigma_p^2 = \mathbf{w}' \mathbf{\Omega} \mathbf{w} \quad (1)$$

where \mathbf{w} is an N -by-1 vector of asset weights, and $\mathbf{\Omega}$ is an N -by- N asset covariance matrix. One form of the well-known solution to this optimization problem is

$$\mathbf{w}_{MV} = \sigma_{MV}^2 \mathbf{\Omega}^{-1} \mathbf{1} \quad (2)$$

where $\mathbf{1}$ is N -by-1 vector of ones. As a practical matter, $\sigma_{MV}^2 = 1 / \mathbf{1}' \mathbf{\Omega}^{-1} \mathbf{1}$ in Equation 2 is simply a scaling parameter which enforces the budget constraint that asset weights sum to one.

The Maximum Diversification objective is to maximize the ‘‘Diversification Ratio’’

$$D_p = \frac{\mathbf{w}' \boldsymbol{\sigma}}{\sqrt{\mathbf{w}' \mathbf{\Omega} \mathbf{w}}} \quad (3)$$

where $\boldsymbol{\sigma}$ is an N -by-1 vector of asset volatilities; the square-root of the diagonal terms of $\mathbf{\Omega}$. Equation 3 has the form of a Sharpe ratio, where the asset volatility vector, $\boldsymbol{\sigma}$, replaces the expected excess returns vector. Using the well-known solution to the tangent (i.e., maximum Sharpe ratio) portfolio with this substitution gives the optimal Maximum Diversification weight vector as

$$\mathbf{w}_{MD} = \left(\frac{\sigma_{MD}^2}{\sigma_A} \right) \mathbf{\Omega}^{-1} \boldsymbol{\sigma}. \quad (4)$$

where σ_A is the weighted average asset risk. The key conceptual difference between the Minimum Variance solution in Equation 2 and the Maximum Diversification solution in equation 4 is not the scaling parameters (first term in each), but the post multiplication of the inverse covariance matrix by the asset risk vector.

The risk-budgeting interpretation of Risk Parity is based on the restatement of Equation 1 as a double sum

$$\sigma_P^2 = \sum_{i=1}^N w_i \sum_{j=1}^N w_j \Omega_{i,j}. \quad (5)$$

where $\Omega_{i,j}$ are the elements in the asset covariance matrix $\mathbf{\Omega}$. A portfolio has the property of Risk Parity if the total (rather than marginal) risk contribution is the same for all assets

$$\frac{w_i \sum_{j=1}^N w_j \Omega_{i,j}}{\sigma_P^2} = \frac{1}{N}. \quad (6)$$

An equivalent “portfolio beta” interpretation is that Risk Parity is achieved when weights are equal to the inverse of their beta with respect to the final portfolio,

$$w_i = \frac{1}{N \beta_{i,P}} \quad (7)$$

where

$$\beta_{i,P} = \frac{\sum_{j=1}^N w_j \Omega_{i,j}}{\sigma_P^2}. \quad (8)$$

Note that the asset beta with respect the Risk Parity portfolio defined in Equation 8 is not the same as β_i , the notation we use for the beta of the asset with respect to the common risk factor.

In this study, we use a single-factor risk model for the asset covariance matrix, a common simplifying assumption in portfolio theory, but also describe the results for a more general K -factor model. Using matrix notation, the N -by- N asset covariance matrix in a single-factor model is

$$\mathbf{\Omega} = \mathbf{\beta}\mathbf{\beta}'\sigma_F^2 + \text{Diag}(\sigma_\varepsilon^2) \quad (9)$$

where $\boldsymbol{\beta}$ is an N -by-1 vector of risk-factor loadings, σ_F^2 is the risk factor variance, and $\boldsymbol{\sigma}_\varepsilon^2$ is an N -by-1 vector of idiosyncratic risks. Using the Matrix Inversion Lemma, the inverse covariance matrix in the single-factor model is analytically solvable

$$\boldsymbol{\Omega}^{-1} = \text{Diag}(1/\boldsymbol{\sigma}^2) - \frac{(\boldsymbol{\beta}/\boldsymbol{\sigma}^2)(\boldsymbol{\beta}/\boldsymbol{\sigma}^2)'}{\frac{1}{\sigma_F^2} + (\boldsymbol{\beta}/\boldsymbol{\sigma}^2)'\boldsymbol{\beta}} \quad (10)$$

and can be substituted into the optimization solutions in Equations 2 and 4.

With this substitution the individual optimal weights in the long-only Minimum Variance solution can be expressed as

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2} \left(1 - \frac{\beta_i}{\beta_L} \right) \text{ for } \beta_i < \beta_L \text{ else } = 0 \quad (11)$$

where β_L is a “long-only threshold” beta that cannot be exceeding in order for an asset to be in solution. The long-only threshold beta is a function of final portfolio risk estimates

$$\beta_L = \frac{\sigma_{MV}^2}{\beta_{MV} \sigma_F^2} \quad (12)$$

where β_{MV} is the risk-factor beta of the long-only Minimum Variance portfolio. As a more practical matter, the threshold beta can be calculated from summations of individual asset risk statistics that come into solution,

$$\beta_L = \frac{\frac{1}{\sigma_F^2} + \sum_{\beta_i < \beta_L} \frac{\beta_i^2}{\sigma_{\varepsilon,i}^2}}{\sum_{\beta_i < \beta_L} \frac{\beta_i}{\sigma_{\varepsilon,i}^2}} \quad (13)$$

Equation 13 is a more practical than Equation 12 in the sense that assets can be sorted in order of ascending factor beta, and then compared to the summations until individual asset beta exceeds

the long-only threshold. Together with the fact that portfolio risk is simply a scaling factor in Equation 11, numerical search routines are not required to find the set of assets and their optimal weights for long-only Minimum Variance portfolios. Scherer (2011) provides analytic work that is similar in form to Equation 11, but for unconstrained long/short portfolios.

For a K -factor risk model, the general unconstrained solution to the Minimum Variance portfolio is

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2} \left(1 - \sum_{k=1}^K \frac{\beta_{k,i}}{\beta_k} \right) \quad (14)$$

where β_k is a portfolio (i.e., not asset specific) parameter, calculated as

$$\beta_k = \frac{\sigma_{MV}^2}{\sum_{l=1}^K \beta_{MV,l} V_{k,l}}. \quad (15)$$

In Equation 15, $\beta_{MV,l}$ is the portfolio beta with respect to the l^{th} risk-factor and $V_{k,l}$ is an element of the K -by- K factor covariance matrix. The K -factor solution in Equation 14 also has a long-only *constrained* version, although the criterion for inclusion in the portfolio involves multiple sorts and is therefore more complicated to calculate.

Substitution of the inverse covariance matrix from Equation 10 into Equation 4, and some algebra, gives a relatively simple solution for individual weights in the Maximum Diversification portfolio

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2} \frac{\sigma_i}{\sigma_A} \left(1 - \frac{\rho_i}{\rho_L} \right) \text{ for } \rho_i < \rho_L \text{ else } = 0 \quad (16)$$

where σ_i is total asset risk, ρ_i is the correlation coefficient between the asset and the common risk factor, and ρ_L is a “long-only threshold” correlation that cannot be exceeded for an asset to be in solution. The long-only threshold correlation can be described as a function of final portfolio risk estimates

$$\rho_L = \frac{\sigma_{MD}^2}{\sigma_A \beta_{MD} \sigma_F} \quad (17)$$

where β_{MD} is the factor beta and σ_A is the average asset risk, respectively. As a more practical matter, the long-only threshold correlation can be calculated from summations of individual asset correlations that come into solution,

$$\rho_L = \frac{1 + \sum_{\rho_i < \rho_L} \frac{\rho_i^2}{1 - \rho_i^2}}{\sum_{\rho_i < \rho_L} \frac{\rho_i}{1 - \rho_i^2}}. \quad (18)$$

Equation 18 is a more practical than Equation 17 in the sense that assets can be sorted in order of ascending factor correlation, and then compared to the summations until the individual asset correlation exceeds the long-only threshold. Note that assets sorted by correlation will be in a somewhat different order than sorting by beta due to differences in idiosyncratic risk. Together with the fact that final portfolio risk divided by average asset risk is simply a scaling factor in Equation 16, numerical search routines are not required to find the set of assets and their optimal weights for long-only Maximum Diversification portfolios. Carvalho, Lu, and Moulin (2011) provide analytic work that produces a form similar to Equation 14, but for unconstrained long/short portfolios.

For a K -factor risk model, the general unconstrained solution to the Maximum Diversification portfolio is

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2} \frac{\sigma_i}{\sigma_A} \left(1 - \sum_{k=1}^K \frac{\rho_k}{\rho_k} \right) \quad (19)$$

where ρ_k is a portfolio (i.e., not asset specific) parameter

$$\rho_k = \frac{\sigma_{MD}^2}{\sigma_A \sum_{l=1}^K \beta_{MD,l} V_{k,l}}. \quad (20)$$

In Equation 20, $\beta_{MD,i}$ is the portfolio beta with respect to the l^{th} risk-factor and $V_{k,l}$ is an element of the K -by- K factor covariance matrix. The K -factor solution in Equation 19 has a long-only *constrained* version, although criterion for inclusion in the portfolio involves multiple sorts and is therefore more complicated to calculate.

Risk Parity portfolio weights under a single-factor risk-model can be derived by substituting the individual covariance matrix terms of Equation 9 into Equation 6, and applying the quadratic formula. Using the “positive root” in that formula gives

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2} \left[\left(\frac{\beta_i^2}{\gamma^2} + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2} \right)^{1/2} - \frac{\beta_i}{\gamma} \right] \quad (21)$$

where

$$\gamma = \frac{2 \sigma_{RP}^2}{\beta_{RP} \sigma_F^2} \quad (22)$$

is a constant term across asset weights that includes β_{RP} , the beta of the Risk Parity portfolio with respect to the risk factor. Unlike the equations for individual weights in the Minimum Variance and Maximum Diversification portfolios, Equation 21 cannot be used to inform a simple asset sort to calculate weights because final portfolio terms are embedded in the equation, not simply part of the scaling factor. However, Equation 21 does inform a quick numerical routine for even large investable sets. The numerical process starts with an equally-weighted portfolio, and then iteratively calculates the portfolio parameters (β_{RP} and σ_{RP}) and then asset weights according to Equation 21, until weights equal one over N times their beta with respect to the final portfolio, $\beta_{i,RP}$, in accordance with Equation 7. We also note that the quadratic formula allows for a “negative root” that is real. However, the positive root, specified in Equation 21, is the only one that maintains the budget constraint that the asset weights sum to one.

For a general K -factor risk model, the solution to the Risk Parity portfolio is

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2} \left[\left(\left(\sum_{k=1}^K \frac{\beta_{k,i}}{\gamma_k} \right)^2 + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2} \right)^{1/2} - \sum_{k=1}^K \frac{\beta_{k,i}}{\gamma_k} \right] \quad (23)$$

where

$$\gamma_k = \frac{2\sigma_{RP}^2}{\sum_{l=1}^K \beta_{RP,l} V_{k,l}}. \quad (24)$$

The general K -factor Risk Parity portfolio is long-only by definition and can be used to inform a quick calculation routine similar to the one described above for the single-factor model.

Under the single-factor model, the implications of differences in beta and idiosyncratic risk on the relative magnitude of the weights in the Minimum Variance and Maximum Diversification portfolios is fairly evident from the form of Equations 11 and 16. The more complex form for Risk Parity portfolio weights in Equation 21 makes the impact of the different asset risk parameters less apparent, motivating formal calculus. The partial derivative of the Risk Parity asset weight in Equation 21 with respect to asset beta is

$$\frac{\partial w_{RP,i}}{\partial \beta_i} = \frac{-w_{RP,i}}{\gamma \left(\frac{\beta_i^2}{\gamma^2} + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2} \right)^{1/2}} \quad (25)$$

and thus always negative because weights in the long-only solution are positive, as is the square-root term in the numerator of Equation 25. The derivative is “partial” in the sense that Equation 25 applies to the magnitude of asset weights compared to each other in a risk-balanced set. A full derivative would be more complex because a change in the risk parameter for any single asset will change the Risk Parity weights for the entire set.

The functional form of Equation 21 (i.e., squared terms and square-roots) for Risk Parity portfolios indicates that weights asymptotically approach zero with high beta, and increase with low beta, including negative betas if any. Indeed, under the assumption of homogenous idiosyncratic risk, Risk Parity weights form the positive side of a non-rectangular hyperbola,

centered at the origin, with asymptotes of slope zero and $-(\beta_{RP} \sigma_F^2) / \sigma_\varepsilon^2$. For example, an asset with a common factor beta of zero would have a weight of $\sigma_{RP} / \sigma_\varepsilon \sqrt{N}$.

Assets with a large negative beta with respect to the single risk factor, β_i , may have negative beta with respect to the final portfolio, $\beta_{i,RP}$. However, negative weight assets were likely not envisioned by those who formalized the Risk Parity condition. Specifically, an asset with a small positive $\beta_{i,RP}$ would have a relatively large positive weight according to the definition in Equation 7. But a slightly different asset with a small negative $\beta_{i,RP}$ would have a large negative weight, a nonsensical result for an asset with the potential to hedge the single risk factor. The lower limit for β_i on any single asset that still allows for a Risk Parity portfolio involving all investable assets is complex, although it can be shown that under the assumption of homogenous idiosyncratic risk, the lower limit is $-\sigma_\varepsilon^2 / (\beta_{RP} \sigma_F^2)$, the arithmetic inverse of the hyperbolic asymptotic slope. The number of iterations for convergence in the numerical process specified above for Risk Parity portfolios increases for investable sets that have betas which approach that limit, although betas of that magnitude were not encountered anywhere in the 516 months (1968 to 2010) for the 1000 largest U.S. stocks.

In addition to the single factor model, other useful simplifications of a general covariance matrix are the specialized case of homogeneous idiosyncratic risk, and the Constant Correlation model. If one assumes that that idiosyncratic risk is the same for all assets, $\sigma_{\varepsilon,i}^2 = \sigma_\varepsilon^2$, then the risk of any portfolio under a single-factor model is

$$\sigma_p^2 = \beta_p^2 \sigma_F^2 + \sigma_\varepsilon^2 \sum_{i=1}^N w_i^2 \quad (26)$$

where the portfolio beta with respect to the risk factor is by definition, $\beta_p = \sum_{i=1}^N w_i \beta_i$. This motivates a measure of portfolio concentration called “Effective N” (Strongin, Petsch, and Sharenow, 2000),

$$N_{Eff} = \frac{1}{\sum_{i=1}^N w_i^2} \quad (27)$$

which can be interpreted as the number of equally-weighted stocks that would provide the same diversification benefits as the actual portfolio. Effective N ranges from 1 (the entire portfolio is invested in a single asset) to N (an equally-weighted portfolio). Most measures of concentration, for example the Herfindahl Index in general economics, are also related to the sum of squared weights.

The Constant Correlation model of Elton and Gruber (1973) allows for heterogeneous risk across assets, but the same pair-wise correlation between all assets pairs. In terms of the single-factor model, this equates to assuming that asset correlations to the common risk factor are all the same. Under the Constant Correlation model, it can be shown that the Maximum Diversification portfolio includes all assets (i.e., portfolio is long-only by design) and equivalent to the Risk Parity portfolio. In addition, under the Constant Correlation model, Maillard, Roncalli, and Teiletche (2010) show that weights are proportional to inverse asset volatility,

$$w_{MD,i} = w_{RP,i} \propto \frac{1}{\sigma_i}. \quad (28)$$

On the other hand, the long-only Minimum Variance does *not* include all the investable assets under the Constant Correlation assumption, and has weights that are a more complex function of asset risk, but generally proportional to inverse variance. For example, under the additional simplifying assumption that the constant correlation value is zero, the weights of the Minimum Variance portfolio are exactly proportional to the inverse asset return variance,

$$w_{MV,i} \propto \frac{1}{\sigma_i^2}. \quad (29)$$

Several important perspectives on the construction of long-only Minimum Variance and Maximum Diversification portfolios can be gleaned from the threshold parameter formulas in Equations 13 and 18. For example, under the simplifying assumption of homogenous idiosyncratic risk, Equation 13 for the long-only threshold beta becomes

$$\beta_L = \frac{\frac{\sigma_\varepsilon^2}{N\sigma_F^2} + \overline{\beta^2}}{\overline{\beta}} \quad (30)$$

where the “bar” over a variable indicates a simple arithmetic average of the risk parameter across assets in solution. The form of Equation 30 suggests that an increase in the size of the asset set, N , reduces the threshold beta, with the limit being the average asset beta squared over average asset beta. A decrease in the threshold beta leads to a lower proportion of assets in solution (e.g., 100 out of 1000 compared to 40 out of 100) according to Equation 11, and thus more relative weight concentration for larger N . Equation 30 also indicates that higher average idiosyncratic risk increases the threshold beta, leading to more assets in solution and less portfolio concentration.

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