Strategic Uncertainty in Asset Markets

Emre Ozdenoren and Kathy Yuan*

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Abstract

We develop a noisy rational expectations equilibrium model of asset prices with informed and uninformed investors. In our model, informed investors affect the asset’s dividend payoff through their investment, thus have coordination incentives. They receive heterogeneous private signals about the asset’s fundamental value and observe its price. The uninformed investors observe only the price. The asset price aggregates private signals and is informative of both fundamental and the coordination component of the asset’s future dividend. Both the informed and the uninformed investors make inferences from the asset price about the level of coordination component. This heightens the information effect relative to the substitution effect and leads to price multiplicity in illiquid markets. In liquid markets, however, asset prices have a limited role in coordinating informed investors’ actions and are not destabilizing. Coordination also leads to increased price volatility. Comparative statics shows price volatility is higher (1) when private signals are more precise, (2) when the noise trading is less or, (3) when the coordination incentive is larger.

*Emre Ozdenoren is at the Economics Department at the University of Michigan and Kathy Yuan is at the Finance Department at the Stephen M. Ross School of Business at the University of Michigan. We thank Marios Angeletos, Bob Barsky, Tilman Borgers, Arvind Krishnamurthy, Alessandro Pavan, seminar participants at Cowles Foundation Workshop on Coordination Games in Yale University, Hong Kong University, Koc University, Michigan State University, Peking University, Sabanci University, Tsinghua University, University of Michigan, for helpful comments.
1. Introduction

It is well established that there is a feedback effect between asset prices and asset values.\footnote{For example, Subrahmanyam and Titman (2001) observe that a higher stock price can make the firm more attractive to potential employees, customers or suppliers, who make decisions contingent on information revealed by prices and whose actions have externalities. Alternatively, the information reflected in the stock price on the firm’s fundamentals may lower the borrowing costs by reducing uncertainty for outside lenders (Sunder (2005)), affect investment decisions (Baker, Stein, and Wurgler (2003); Polk and Sapienza (2002); Chen, Goldstein, and Jiang (2004)), improve the efficiency of capital allocation decisions (Leland (1992); Dow and Gorton (1997); Subrahmanyam and Titman (2001); Dow and Rahi (2003)), or elicit managerial effort through optimal incentive contracts (Holmstrom and Tirole (1993); Bolton and von Thadden (1998)). In our model, as shown later, we utilize the observation that only informed investors play an active role in corporate governance and monitoring and capture one aspect of the feedback effect, which is the feedback effect between informed investors’ investment and asset values.} The feedback effect arises when there are complementarities across agents’ economic decisions that affect the asset value and the agents use the asset price to coordinate these decisions. Coordination requires agents to form beliefs (and higher order beliefs) about the actions of other agents. Therefore, these agents face additional uncertainty (beyond the uncertainty about the fundamental value of the asset) which we call strategic uncertainty.

Morris and Shin (1998, 2002, 2003) building on Carlsson and van Damme (1993) present a theoretical framework to study strategic uncertainty. They show that equilibrium is unique when agents observe fundamentals with small enough private noise. However, as observed by Atkeson (2000), there is a difficulty in applying this game-theoretic insight in the asset-pricing context. Asset prices aggregate private information which may facilitate coordination and lead to multiple equilibria in financial markets. This insight is formalized first by Angeletos and Werning (2005) who predict that multiplicity in equilibrium prices will arise robustly.

Our main insight is that learning from asset prices may lead to multiplicity of equilibrium prices only in illiquid asset markets even when investors have an incentive to coordinate. This theoretical finding is in line with the empirical observation that asset markets are remarkably stable, especially in developed market economies, and crashes are a rarity rather than a norm. In contrast, less liquid markets, such as stock markets in emerging economies, are frequently exposed to large price movements.

Strategic uncertainty, however, represents a significant source of volatility in the asset markets even when it is not destabilizing. We find higher volatility in asset prices (1) when the precision of private signals is higher, (2) when the noise trading and hence the noise level of the public signal is small or, (3) when the coordination incentive is larger, especially in illiquid markets. The first two of these comparative statics may seem counter-intuitive as
more information should in general reduce the price volatility. However, in our model a more precise private signal (or a less volatile noise demand) not only is more informative about the fundamental, but also leads to easier coordination. This set of results generates several testable hypotheses on cross-sectional asset prices and volatility. For example, controlling for traditional asset pricing factors (such as Fama-French (1993) three factors), our results indicate illiquid stocks where the coordination incentive is larger should exhibit higher excess volatility, severe over-reaction to public information, more sensitivity to external liquidity shocks, and larger skewness in returns.\footnote{The coordination incentive in our model is captured by the sensitivity to informed investors’ investment. Empirically, it has been shown that the sensitivity to institutional equity acquisition is positively associated with higher returns (Wang (2003)). Our model offers an explanation for such phenomena.}

Our results are derived based on a noisy rational expectations equilibrium (REE) model of asset prices with noisy demand shocks so that prices are not fully revealing of private information, similar to that in Hellwig (1980) and in Grossman and Stiglitz (1980). Like these papers our model has informed investors who receive heterogenous private signals about asset payoffs and uninformed investors who only observe prices. The difference is that in our model informed investors face strategic complementarity in their investment and hence the payoff of an informed agent is not independent of the actions of others. Through asset prices, uninformed investors become less asymmetrically informed and informed investors coordinate their beliefs and investment.

In our model, uninformed investors do not have coordination incentives themselves. Nevertheless, they observe prices and from prices they can infer the likelihood of coordination among informed investors. Our model allows for this possibility which is unique in the literature on coordination and asset prices and it accords well with the reality. Empirically, since the late 1980s, around 85% of total market capitalization in the U.S. is held by small individual investors and the rest by the large institution investors. Institutional investors are known to acquire information and play an active role in corporate governance and monitoring, thus their investment has a feedback effect.\footnote{For example, Hartzell and Starks (2003) find that the institutional ownership affects executive compensation contracts. Bharath and Yuan (2006) find a positive relationship between the institutional ownership and lower bank borrowing costs.} They are the informed investors in our model. Individual investors, who are mostly uninformed, do not play a significant role in corporate governance or monitoring, thus their investments are unlikely to affect the value of the stock. Furthermore, it is difficult for them to coordinate their investments since they are large in numbers and their investments are small. They are the uninformed investors in
our model.

By characterizing strategic uncertainty explicitly, we pinpoint sources of volatility and multiplicity in equilibrium. Typically, asset prices in REE settings have two roles. One is substitutional, which is to clear the market, and the other is informational, which is to provide information regarding the value of fundamentals.\(^4\) The substitution effect of asset prices dominates the information effect in a typical Grossman-Stiglitz setup with one risky asset and information asymmetry, which implies unique equilibrium (see the discussion of non-linear REE in Yuan (2005)). Multiplicity arises when the information effect overwhelms the substitution effect. This happens, for example, when there is an additional uncertainty (other than the payoff uncertainty).\(^5\) In asset markets with both fundamental and strategic uncertainties, we find that the information effect of asset prices is heightened because prices are also indicative of the likelihood of coordination among informed investors. The information effect now has two components: coordination and fundamental. By analyzing these two components, we find that the source of volatility and multiplicity is not the informed investors’ inference about the fundamental component but rather about the coordination component. Even if private signals are close-to fully revealing of the fundamental, in a liquid market, it is difficult for an individual informed investor to form beliefs about other informed investors’ actions. Prices are destabilizing only in illiquid markets because in these markets, sharp forecasts of other informed investors’ actions are possible.

Moreover, by modeling uninformed investors’ behavior and allowing them to predict informed investors’ coordination behavior through publicly observable signals such as prices, we find the information effect of asset prices is further strengthened. This additional effect can lead to price multiplicity, even when informed investors demand for the asset is downward sloping.

Closely related papers are Angeletos and Werning (2005), Hellwig, Mukherji, and Tsyvinski (2005) and Tarashev (2005). Angeletos and Werning study the case where private signals are aggregated in the asset market through price and the price is observable in a separate

\(^4\)The substitution and information effect of equilibrium prices are first used by Admati (1985) to describe the following observation: In REE uninformed agents extract information from prices, and their demand depends on the distribution of equilibrium prices; on the other hand, equilibrium prices must equal supply and demand in the market, the distribution of equilibrium prices also depends on the demand of agent investors.

\(^5\)For example, the additional uncertainty in Gennotte and Leland (1990) is the status of programming trading, in Barlevy and Veronesi (2003) it is the wealth and short sales constraints, and in Yuan (2005) it is the status of borrowing constraints. In these cases, the source of volatility and multiplicity is due to the non-linear inference problem faced by uninformed investors.
coordination game. Hellwig, Mukherji, and Tsyvinski (2005) and Tarashev (2005) consider the coordination problem within the currency market, through the dependence of the central banks' devaluation decision on interest rates.\textsuperscript{6} We are similar to the latter two papers in embedding the coordination motive into a primary market. Yet our paper differs in several respects. In our paper, the coordination motive is introduced through strategic complementarity in investment decisions of informed investors. Moreover, we focus on modeling strategic uncertainty in asset markets and examining the properties of asset prices in relation to strategic uncertainties. By contrast this literature is interested in analyzing speculative currency attacks and the role of policy makers such as central banks in the currency market. Furthermore, in our model multiplicity in equilibrium asset prices occurs not just from the strategic interaction among informed investors, but also through the non-linear inference problem faced by uninformed investors. Finally, we differ in our interpretation of the information aggregation role of asset prices. In particular, we emphasize that the limited role of prices in destabilizing the asset market.\textsuperscript{7}

The remainder of the paper is organized as follows. In Section 2, the model setup for an economy with one risky asset is first developed. In Section 3, we study the coordination problem among heterogeneously informed investors. In this setup informed investors submit only market orders for the risky asset and they do not observe nor make inferences based on asset prices. In Section 4, heterogeneously informed investors observe an additional signal for the asset’s terminal payoff, that is, the asset price. They are allowed to submit a demand curve for the risky asset. We present the solution for their equilibrium belief and actions and analyze the properties of equilibrium prices. In Section 5, uninformed investors also make inferences based on asset prices. We explore corresponding equilibrium properties. Section 6 concludes. All proofs are collected in the Appendix.

\section{The Model Setup}

In this section we describe the model. We incorporate multiple types of investors and state-trade dependent terminal value to study the strategic interaction among informed investors

\textsuperscript{6}See Hellwig, Mukherji, and Tsyvinski (2005) for discussions on the results in Tarashev (2005).

\textsuperscript{7}It is an emerging and active field of research to apply the insight in Carlsson and van Damme (1993) and Morris and Shin (1998, 2002, 2003) to financial markets. Examples include (but are not limited to) Goldstein and Pauzner (2005); Morris and Shin (2005); Chamley (2003); Allen, Morris, and Shin (Forthcoming); Platin (2004); and Ozdenoren and Yuan (2005).
and the implication of strategic uncertainty for the asset prices. We first detail the main features of the assets in the model. We then introduce agents and the information structure.

**Assets**

We consider a one-period economy with two assets, a riskless bond and a risky asset. For simplicity, we use the bond as the numeraire. Hence, its price remains at one and the risk-free rate is zero. The risky asset can be thought of as a common stock, an equity claim on a firm. The risky asset has an aggregate supply of $M$ where $M > 0$ and a risky terminal payoff which consists of two components, $V + \nu$, and is determined by

$$V = \tilde{V} + \nu, \quad \nu = \sigma_v \epsilon_v, \quad \nu = f(X, \theta).$$

(1)

We assume that $\tilde{V} = 0$, $\sigma_v$ is a positive constant, and $\epsilon_v$ is a standard normal (with zero mean and unit volatility) and independently and identically distributed (i.i.d.). The first risky component of the dividend payoff, $V$, can be regarded as the payoff from the firm’s normal, e.g., bricks-mortar, operations, where no R&D is needed. The second risky component of dividend payoff, $f(X, \theta)$, comes from the stochastic payoff of the firm’s new technology innovation. Here $\theta$, which is drawn from the uniform distribution over the real line, is the fundamental value of the innovation and $X$ is the amount invested in the risky asset by informed investors. We assume that $f(X, \theta)$ is positively related to both $\theta$ and $X$. This assumption captures the strategic complementarity among informed investors: The value of the equity is higher when more informed investors are investing in it. $V + \nu$ is realized and revealed at the terminal date.

The assumption on $f(X, \theta)$ is motivated by the feedback effect of stock price to the firm value. The feedback effect is based on the hypothesis that stock prices reflect information about firms’ fundamentals. In our setup, the feedback effect is captured through firm value increasing in the amount invested by informed investors. This is because only informed investors’ demand contains information regarding firms’ fundamentals in our model, a sufficient statistic for price informativeness is the amount of investment made by informed investors.\footnote{We derive in the Appendix a particular functional form for $f(X, \theta)$ in a model where managers learn about the quality of the firm’s underlying project from the trading in the asset market and choose their optimal effort level accordingly. A similar argument is used in motivating the empirical tests on feedback.}
This observation is supported by empirical evidence. By analyzing a broad cross-section of NYSE-listed stocks between 1983 and 2003, Boehmer and Kelley (2005) find that stocks with greater institutional ownership increases the information efficiency of share prices. Hartzell and Starks (2003) provide evidence that institutional investors affect the incentive contracts of managers. Bharath and Yuan (2006) find that greater institutional ownership reduces borrowing costs.

**Investors**

There are two types of investors in this economy: long-term uninformed investors and informed investors.

Long-term uninformed investors, of a measure-$w$ continuum, are mean-variance investors with the same risk aversion parameter, $\rho$. Hence they have the following aggregate demand curve for the risky asset:

$$L(P) = w\left(\frac{E(V) - P}{\rho \text{var}(V)}\right).$$

(2)

According to this demand curve, long-term uninformed investors provide liquidity in the market. When the price falls below the expected fundamental value that is unrelated to the strategic complementarity, long-term uninformed investors will buy the asset.\(^{12}\) The slope of this demand curve, $w/(\rho \text{var}(V))$, which we denote by $1/\lambda$, measures the liquidity provided by the long-term uninformed investors.

Informed investors is of a measure-one continuum of informed investors, indexed by $i \in [0, 1]$. They have access to an information-production technology. This technology enables each informed investor to acquire a noisy private signal ($s_i$) at time 0 about $\theta$, the potential payoff of the new technology,

$$s_i = \theta + \sigma_s \epsilon_i,$$

(3)

where $\epsilon_i$ is uniformly distributed on $[-1, 1]$.\(^{13}\) We denote the density of this distribution by effects in Chen, Goldstein, and Jiang (2004).

\(^{12}\)Essentially we assume that long-term investors are uninformed investors who are either unaware of the strategic complementarity component, or expect not to be compensated from holding the idiosyncratic risk, i.e., $\theta$, of a particular firm. This allows us to capture the fact that long-term investors are uninformed liquidity providers and enables us to focus on analyzing the strategic interaction among informed investors first. In Section 5, we relax this assumption and study the case when uninformed investors are aware of the strategic complementarity component of the dividend payoff and make inferences through market clearing prices.

\(^{13}\)The assumption on the noise term being uniformly distributed is for the ease of exposition and is not
Conditional on $\theta$, the signals are i.i.d. across informed investors. These informed investors are risk neutral and seek to maximize their expected profit. We further assume that each informed investor is restricted to trade $x(i) \in [0, z]$, where $z$ is a fixed number and $z \geq 1$. This position limit can be due to limited capital and borrowing constraints faced by informed investors.\(^{14}\) We denote the total demand from informed investors by $X = \int_0^1 x(i) \, di$.

We denote an informed investor’s utility from buying $k \in [0, z]$ units of the stock by $U(k, X, \theta, P)$. Hence $U(k, X, \theta, P) = k(f(X, \theta) - P)$ where $f(X, \theta)$ is the dividend payoff from the stock and $P$ is the price of the stock.

Note that, because of risk neutrality, an informed investor would either invest up to the position limit, $z$, or not invest at all. Therefore, the total demand, $X$, depends on the fraction of informed investors investing in the stock and the position limit, $z$.

### 3. Benchmark Model: Only Private Information

As a benchmark, we first study the case where informed investors are only allowed to submit market orders and do not observe (nor rationally expect) the equilibrium price. Their information set contains only a private signal $s$, so their investment decision is only a function of $s$. In what follows, we first introduce the equilibrium concept and then solve the model. Finally, we illustrate equilibrium properties related to strategic uncertainties using a linear functional form of $f(X, \theta)$.

**Definition 1** An equilibrium is a price function, $P(\theta)$, strategies, $\pi(s_i) : \mathbb{R} \rightarrow \{0, 1\}$, for each informed investor where $\pi_i(s_i) = 1$ if she buys the stock and 0 otherwise, and the corresponding aggregate demands such that:

- For informed agent $i$, $\pi(s_i) \in \arg\max_{\pi} E[U(\pi, X(\theta), \theta, P)|\mathcal{F}_i]$, where $\mathcal{F}_i = \{s_i\}$ is informed investor $i$’s information set.

- The aggregate demand from informed agents, $X(\theta)$, is given by $\frac{z}{\sigma_s} \int_s \pi(s)h\left(\frac{s-\theta}{\sigma_s}\right)ds$.

- Uninformed long-term investors demand, $L(P)$, is given by $w(E(V) - P)/(\rho \text{Var}(V))$.

- The market clearing condition is satisfied: $X(\theta) + L(P) = M$.

\(^{14}\)The specific size of this position limit on asset holdings is not crucial for our results. What is crucial is that informed investors cannot take unlimited positions; if they do, strategic interaction among informed investors will become immaterial.
Note that the market clearing condition and the simplifying assumption on the mean of the random dividend \( V \) imply

\[
P = \lambda X - \lambda M
\]  

(4)

where \( \lambda = \rho \sigma^2_v / w \). Note that \( \lambda \) also measures the price impact of informed investors’ trades.

Now substituting the equilibrium price in the utility function for the informed investors, we can write their payoff from buying a unit of the stock as \( U(1, X, \theta, P) = f(X, \theta) - \lambda(X - M) \).

To ensure that there are strategic complementarities among informed investors’ decisions to acquire the asset we assume that \( f_X > \lambda \). We will maintain this assumption throughout the paper. With this assumption we obtain the following benchmark result.\(^{15}\)

**Proposition 1** In the game of incomplete information described in Definition 1, there is a unique equilibrium in which each informed agent follows a cutoff strategy of the form:

\[
\pi(s) = \begin{cases} 
1 & \text{if } s \geq \kappa \\
0 & \text{if } s < \kappa 
\end{cases}
\]

Next, for future reference, we illustrate the equilibrium in an example. In this example the risky asset supply \( M = 0 \) and the dividend function is, \( f(X, \theta) = \alpha X + \theta \). Here, \( \alpha \) can be interpreted as a measure of coordination incentive, and action monotonicity implies \( \alpha > \lambda \). The following corollary describes the equilibrium in this example.

**Corollary 1** When \( M = 0 \) and \( f(X, \theta) = \alpha X + \theta \), the unique cutoff in Proposition 1 is given by \( \kappa = (\lambda - \alpha)z/2 \). Informed investors purchase the stock if and only if their private signal is above this cutoff value and the equilibrium price is uniquely determined by

\[
P = \begin{cases} 
0 & \text{if } \frac{\kappa-\theta}{\sigma_x} > 1 \\
\frac{\lambda z}{2} \left( 1 - \frac{\kappa-\theta}{\sigma_x} \right) & \text{if } -1 \leq \frac{\kappa-\theta}{\sigma_x} \leq 1 \\
\lambda z & \text{if } \frac{\kappa-\theta}{\sigma_x} < -1 
\end{cases}
\]

The cutoff value \( \kappa = (\lambda z/2) - (\alpha z/2) \) has a natural intuition. The first term, \( \lambda z/2 \), represents the standard substitution effect. Recall that \( \lambda z/2 = E[P|\mathcal{F}] \) where \( \mathcal{F} = \{\kappa\} \).

When the expected asset price increases by one unit, all else equal an informed investor with

\(^{15}\)In this paper we restrict attention to equilibria in cutoff strategies. Under further mild technical assumptions Morris and Shin (2003) prove a stronger result. They show that as the private signals become more precise, a unique cutoff strategy survives iterative elimination of strictly dominated strategies, and all the other strategies that survive elimination differ from this cutoff strategy only in a small neighborhood around the cutoff.
the cutoff signal would not purchase the stock unless her private signal also increases by one unit. The second term, $\alpha z/2$, captures the coordination benefit among informed investors. Since coordination yields a positive payoff at the rate $\alpha$ and expected amount of the risky asset purchased is $z/2$, the informed investor with the cutoff signal is willing to purchase the stock even though her signal is $\alpha z/2$ less than the price.

In the next section, we extend the benchmark model and we allow the informed investors to learn from asset prices in addition to their private signals.

4. Asset Prices as Public Signals for Informed Investors

In this section, we consider the case where informed agents submit demand curves for the risky stock, which can be considered as continuous limit orders. In addition to the private signal, $s$, informed investors’ information set now contains a public signal, the market-clearing price, $P$. To prevent the market-clearing price from fully revealing, we introduce noise in the information aggregation process by assuming there are noise traders in the market as in Grossman and Stiglitz (1980) and a host of other models in the asymmetric information literature (e.g., Kyle (1985); Wang (1993); Vayanos (2001); Yuan (2005)). More specifically, we assume that noise traders’ demand is $\sigma y$, where $\sigma > 0$ and $y$ is a standard normal random variable and is independent of $\epsilon_v$, $\theta$, and $\epsilon_i$ for all $i$.\(^\text{16}\) In what follows, we will first present the equilibrium solution for the case where the dividend payoff, $f(X, \theta)$, is of a general form and then illustrate intuitions by analyzing the explicit solution for the case where the dividend payoff is linear in $X$ and $\theta$, i.e., $f(X, \theta) = \alpha X + \theta$.

4.1. General Dividend Payoff Function

Public signals such as market-clearing asset prices are different from the exogenous public signals. The precision of the information conveyed in market clearing asset prices is endogenously determined and is positively related to the precision of the private information. We first introduce the equilibrium concept corresponding to the endogenous public information in the following definition.

\(^{16}\)The results are essentially the same when we assume that the noise demand follows a double exponential distribution, a fatter tail distribution. The explicit solutions for the double exponential distribution case are available from the authors upon request.
Definition 2 An equilibrium is a price function, \( P(\theta, y) \), strategies, \( \pi(s, P) : \mathbb{R}^2 \to \{0, 1\} \), for each informed investor where \( \pi(s, P) = 1 \) if the investor buys the stock and 0 otherwise, and the corresponding aggregate demands, such that:

- For informed agent \( i \), \( \pi(s, P) \in \arg\max_{\pi} E[U(\pi, X(P, \theta), \theta, P)|\mathcal{F}_i] \), where \( \mathcal{F}_i = \{s, P\} \) is informed investor \( i \)'s information set. The aggregate demand from informed agents, \( X(P, \theta) \), is given by \( \frac{z}{\sigma_s} \int s \pi(s, P) h(\frac{s - \theta}{\sigma_s}) ds \).

- Uninformed long-term investors demand, \( L(P) \), is given by \( w(E(V) - P)/(\rho \text{Var}(V)) \).

- The market clearing condition is satisfied: \( X(P, \theta) + L(P) + \sigma y = M \).

First note that with the addition of noise traders in the economy, the market clearing condition and the simplifying assumption on the mean of the random dividend \( V \) now imply

\[
P = \lambda X + \lambda \sigma y y - \lambda M. \tag{5}
\]

We define a monotone equilibrium (with cutoff strategies) as an equilibrium where informed investors buy the stock if and only if their private signal exceeds a cutoff \( g(P) \). In other words, in a monotone equilibrium informed investor \( i \) buys if and only if \( \epsilon_i \geq \frac{g(P) - \theta}{\sigma_s} \). Therefore, informed investors’ aggregate demand is given by:

\[
X(P, \theta) = \begin{cases} 
0 & \text{if } \frac{g(P) - \theta}{\sigma_s} > 1 \\
\frac{z}{2} \left(1 - \frac{g(P) - \theta}{\sigma_s}\right) & \text{if } -1 \leq \frac{g(P) - \theta}{\sigma_s} \leq 1 \\
\frac{z}{2} & \text{if } \frac{g(P) - \theta}{\sigma_s} < -1
\end{cases} \tag{6}
\]

Combining equations (5) and (6) we see that market clearing prices satisfy

\[
P = \begin{cases} 
\lambda \sigma y y - \lambda M & \text{if } \frac{g(P) - \theta}{\sigma_s} > 1 \\
\frac{z}{2} \left(1 - \frac{g(P) - \theta}{\sigma_s}\right) + \lambda \sigma y y - \lambda M & \text{if } -1 \leq \frac{g(P) - \theta}{\sigma_s} \leq 1 \\
\frac{z}{2} \left(1 - \frac{g(P) - \theta}{\sigma_s}\right) + \lambda \sigma y y - \lambda M & \text{if } \frac{g(P) - \theta}{\sigma_s} < -1
\end{cases} \tag{7}
\]

From Eq. (7), we observe that the market clearing prices are uninformative about \( \theta \) when \( \frac{g(P) - \theta}{\sigma_s} > 1 \) or when \( \frac{g(P) - \theta}{\sigma_s} < -1 \). However, in the intermediate region when \(-1 \leq \frac{g(P) - \theta}{\sigma_s} \leq 1 \), we have

\[
t \equiv \left(\frac{2\sigma_s}{\lambda z} P + \frac{2\sigma_s}{z} M + g(P) - \sigma_s\right) = \theta + \frac{2\sigma_s \sigma y}{z} y, \tag{8}
\]

which is observable to informed investors and is uncorrelated with their private signals conditional on \( \theta \). Hence, \( t \) is a sufficient statistics for the information in \( P \) in this region. It is
distributed normally with a mean of $\theta$ and a standard deviation of $2\sigma_x\sigma_y/z$. Note that the precision of $t$, or the market clearing price, $P$, as a public signal for the fundamentals, $\theta$, is endogenously determined. It decreases with the variance of exogenous private signal, $\sigma_x^2$, the variance of noise traders’ demand, $\sigma_y^2$, and increases with the size of informed investors’ position limit, $z$. Given the information conveyed in $P$, we can solve for informed investors’ cutoff strategy $g(P)$ and through market clearing condition solve for the equilibrium price(s). The following proposition provides the equilibrium characterization.

**Proposition 2** Consider the game with incomplete information described in this section and its equilibrium as described in Definition 2. In this game,

- there exists a unique monotone equilibrium, that is, there is a unique function $g : \mathbb{R} \to \mathbb{R}$ such that the equilibrium strategies of informed investors are given by $\pi(s, P) = 1$ if $s \geq g(P)$ and 0 otherwise;

- in this monotone equilibrium, given a market clearing price $P$, informed investors’ aggregate demand, $X(P, \theta)$, is uniquely characterized by Eq. (6) and uninformed long-term investors’ demand is given uniquely by $L(P) = -P/\lambda$ where $\lambda = (wE(V))/(\rho \text{Var}(V))$;

- and the equilibrium price $P(\theta, y)$ is an element of the set of $P$ that satisfies Eq. (7).

Proposition 2 shows that for a given equilibrium price, informed investors’ strategies are uniquely determined on the equilibrium path. This is a standard feature of rational expectations models: A given price must lead to a unique realization of demand from informed investors. However, given the equilibrium strategy of informed investors, equilibrium prices may not be unique in our model. Multiplicity of equilibrium prices arises whenever the aggregate demand from both informed and uninformed investors, $X(P, \theta) + L(P)$, has a backward-bending region where it is increasing in price (or equivalently whenever Eq. (7) has multiple solutions).\(^{17}\)

To understand the intuition behind Proposition 2, in the next subsection, we provide an example where the dividend payoff is linear in $X$ and $\theta$. In this example, we explicitly solve for the equilibrium strategies and provide a necessary and sufficient condition for the existence of a unique market clearing price.

\(^{17}\)Since, by Proposition 2, $L(P)$ is always decreasing, it is necessary but not sufficient for multiplicity to arise that the demand from informed investors $X(P, \theta)$ have a backward bending region.
4.2. A Linear Dividend Payoff Function

The next lemma provides the solution for informed investors’ equilibrium strategies when
\( f(X, \theta) = \alpha X + \theta \).

**Lemma 1** The equilibrium cutoff strategy, \( g(P) \), when the dividend payoff function is \( f(X, \theta) = \alpha X + \theta \), is unique and is characterized by the following equation
\[
g(P) = P + \sigma_s - \left( \alpha + \frac{2\sigma_s}{\lambda} \right) \left( M + \frac{P}{\lambda} - z \right) = g(P).
\]

To understand this result intuitively, it is helpful to go through the following thought experiment. Consider the informed investor who receives the cutoff signal, \( \hat{s} = g(P) \). She must be indifferent between investing and staying out, which implies
\[
E[\alpha X + \theta | F] - P = 0,
\]
where \( F = \{ \hat{s}, P \} \). Given her estimate of the risky asset held by informed investors, \( E[X|F] \), her estimate of \( E[\theta|F] \) can be computed using Eq. (6) as,
\[
E[\theta|F] = \sigma_s \left( \frac{2E[X|F]}{z} - 1 \right) + \hat{s}.
\]

Substituting Eq. (11) to Eq. (10) and using the market clearing condition, we obtain
\[
g(P) = \hat{s} = P + \sigma_s - \left( \alpha + \frac{2\sigma_s}{\lambda} \right) E[X|F] = P + \sigma_s - \left( \alpha + \frac{2\sigma_s}{\lambda} \right) \left( M + \frac{P}{\lambda} - E[\sigma y|F] \right)
\]
which is Eq. (9) in the previous lemma.

Next we analyze how informed investors make inferences from prices by examining the price sensitivity of their cutoff strategy which is given by:
\[
\frac{\partial g(P)}{\partial P} = 1 - \frac{\alpha \frac{\partial E[X|F]}{\partial P}}{1} + \frac{\partial E[\theta|F]}{\partial P}.
\]

Terms in Eq. (13) illustrate the substitution and information effects of prices on the informed investors’ optimal investment strategies. To see this, suppose that the asset price increases by one unit. Normally informed investors would not purchase the risky asset unless their signal also increases by one unit. This is the standard substitution effect. However, the increase in price may also signal a higher likelihood of coordination in investment (i.e.,
the coordination component of the dividend payoff) and better fundamentals (i.e., the fundamental component of the dividend payoff). As a result, informed investors may purchase the risky asset with a lower signal as price increases due to this information effect, i.e., \( g(P) \) is decreasing in \( P \).

Let \( \Lambda(M + P/\lambda, z, \sigma_y) = \lambda \partial E [\sigma_y | \mathcal{F}] / \partial P \). Using this notation we can rewrite Eq. (13) as,

\[
\frac{\partial g(P)}{\partial P} = 1 - \alpha \left( 1 - \frac{\Lambda(M + P/\lambda, z, \sigma_y)}{\lambda} \right) - \frac{2\sigma_s}{z} \left( 1 - \frac{\Lambda(M + P/\lambda, z, \sigma_y)}{\lambda} \right). \tag{14}
\]

From Eq. (14) we can characterize a necessary and sufficient condition under which the information effect overwhelms the substitution effect. This condition is provided in the following proposition along with limiting results.

**Proposition 3** The following are equivalent:

(i) the information effect is larger than the substitution effect at \( P \) for informed investors;

(ii) the cutoff function \( g(P) \) is decreasing at \( P \);

(iii) \((\alpha + 2\sigma_s/z) \left( 1 - \Lambda(M + P/\lambda, z, \sigma_y) \right) > \lambda\), where \( 0 < \Lambda(M + P/\lambda, z, \sigma_y) \leq 1 \).

Moreover, the function \( \Lambda(M + P/\lambda, z, \sigma_y) \) is increasing in \( \sigma_y \) and approaches 0 as \( \sigma_y \) approaches to \( 0 \) when \( -\lambda M \leq P \leq -\lambda(M - z) \). It approaches 1 as \( \sigma_y \) approaches to \( \infty \).

From Proposition 3 we observe that the information effect overwhelms the substitution effect when \( \alpha \) is large enough, \( \lambda \) is small enough, or \( \sigma_s \) is large enough. These conditions are intuitive. When \( \alpha \) is bigger, the incentive to coordinate is stronger which amplifies the coordination component of the information effect. When \( \lambda \) is smaller, the price impact of informed trades and hence the coordination cost is small, which increases both components of the information effect. When \( \sigma_s \) is larger, the quality of private signals becomes worse, informed investors thus depend more on public information, that is, \( P \), to make inferences on fundamental values, which results in a heightened information effect. This last observation, captured also in Figure 1, is the flip side of the intuition provided in the existing coordination game literature where a more precise private information generally leads to less severe coordination problems since agents depend more on their private information to make investment decisions (Morris and Shin (2003)).
Figure 1: **Equilibrium Cutoff Strategies.** The dotted line, the dashed line, and the solid line in the graph each represents the equilibrium cutoff strategy, \( g(P) \), when \( \sigma_s \) is 20, 60, and 200, respectively. The parameters are chosen to illustrate a stylized example where \( \sigma_y = 4 \), \( \alpha = 2 \), \( z = 20 \), \( \lambda = 1 \), and \( M = 1 \).

The limiting results in Proposition 3 are also intuitive. When \( \sigma_y \) approaches infinity, that is, when the public signal is extremely noisy and uninformative, \( \Lambda(M + P/\lambda, z, \sigma_y) \) approaches one. As a result, \( \partial g(P)/\partial P \) approaches one and the information effect in Eq. (14) disappears. This limiting case is just like the case when informed investors observe only heterogeneous private information and no public information (Corollary 1). In both cases, if the asset price increases by one unit, the threshold signal, \( g(P) \), above which an informed investor is willing to purchase the asset also increases by one unit. This is intuitive since receiving an infinitely noisy public signal is equivalent to not receiving any public signal. When \( \sigma_y \) approaches zero, that is, when the public signal becomes fully revealing of fundamentals, \( \Lambda(M + P/\lambda, z, \sigma_y) \) approaches zero in the region \(-\lambda M \leq P \leq -\lambda(M - z)\)\(^{18}\). As a result \( \partial g(P)/\partial P \) becomes negative and the information effect, described by \((\alpha + 2\sigma_s/z)/\lambda\) is larger than substitution effect.\(^{19}\) Once more this finding is intuitive since fully-revealing prices lead to the common knowledge of the dividend payoff.

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\(^{18}\)Since in the limit there is no noise trading, the market clearing price must fall into the region \(-\lambda M \leq P \leq -\lambda(M - z)\). The upper-bound of this region is attained when all informed investors purchase the asset and the lower-bound is attained when no informed investors purchase the asset.

\(^{19}\)This follows since \( \alpha > \lambda \) by assumption.
Next we examine conditions for uniqueness of equilibrium price. Specifically, we analyze overall aggregate demand from both informed and uninformed investors, \( X(P, \theta) + L(P) \), for the existence of a backward-bending region. It is easy to see from Eq. (6) that the backward bending region can only occur when \( \theta \in [g(P) - \sigma_s, g(P) + \sigma_s] \). In this region \( \partial X(P, \theta)/\partial P = (-z/(2\sigma_s))(\partial g(P)/\partial P) \). Thus, the substitution and information effects are magnified by \( z/(2\sigma_s) \) when aggregated across all informed investors.\(^{20}\) Moreover, since uninformed investors do not learn from prices, their demand only exhibits the substitution effect. They decrease their demand by \( 1/\lambda \) unit when prices increase by one unit, that is, \( \partial L(P)/\partial P = -1/\lambda \). Therefore, the following equation describes the price sensitivity of the overall aggregate demand curve in this region:

\[
\frac{\partial (X(P, \theta) + L(P))}{\partial P} = \left( \begin{array}{c}
\frac{1}{\lambda} \\
\frac{z}{2\sigma_s}
\end{array} \right) + \left( \begin{array}{c}
\frac{z\alpha}{2\sigma_s} \left( \frac{1 - \Lambda(M + P/\lambda, z, \sigma_y)}{\lambda} \right) \\
\frac{1 - \Lambda(M + P/\lambda, z, \sigma_y)}{\lambda}
\end{array} \right)
\]

\[= \left( \begin{array}{c}
(\text{from Uninformed}) \\
(\text{from Informed})
\end{array} \right)
\]

Substitution Effect

\[+ \left( \begin{array}{c}
(\text{Coordination Component}) \\
(\text{Fundamental Component})
\end{array} \right)
\]

\[= \left( \begin{array}{c}
(\text{Information Effect (from Informed)})
\end{array} \right)
\]

Eq. (15) sheds some light on the mechanism that leads to multiplicity of equilibrium prices. Multiple equilibrium prices arise when the aggregate information effect dominates the aggregate substitution effect. This can only happen if the coordination component of the aggregate information effect is significantly large. To see this algebraically, note that the fundamental component of the aggregate information effect is always less than \( 1/\lambda \), the substitution effect exhibit in the uninformed investors’ demand. The key determinant of multiplicity in equilibrium prices is the balance of the coordination component of the information effect and the substitution effect from the informed investors. The following proposition explicitly characterizes the condition for the uniqueness of the equilibrium price.

\(^{20}\)Intuitively, the multiplier \( z/(2\sigma_s) \) arises because with a large \( \sigma_s \) private signals of informed investors become more dispersed and a change in the price would move only a few informed investors from one side of the cutoff to the other, limiting their impact at the aggregate level. Moreover, the price impact of the informed investors as a group is further limited since the investment level of each of them is capped by \( z \). Therefore, with dispersed private signals and constrained investment positions, it is possible that the price sensitivity of the aggregate informed demand is small, even when the price sensitivity of the cutoff strategies may be extremely large (as indicated by graphs in Figures 1 and 2).
together with the limiting results.

**Proposition 4** The following are equivalent:

(i) The equilibrium price is unique;

(ii) the aggregate information effect is smaller than the aggregate substitution effect;

(iii) \((\alpha + 2\sigma_s/z)\Lambda(M + P/\lambda, z, \sigma_y) > \alpha - \lambda\) for all \(P\).

Moreover, as \(\sigma_y\) approaches to zero, there are realizations of noise trade such that multiple equilibrium prices would occur. As \(\sigma_y\) approaches to \(\infty\), on the other hand, there is always a unique equilibrium price. As \(\sigma_s\) approaches to \(\infty\), the equilibrium price is always unique. As \(\sigma_s\) approaches to zero, however, the equilibrium price is unique if and only if \(\sigma_y \geq \overline{\sigma}_y\) where \(\overline{\sigma}_y\) satisfies \(\lambda - \alpha(1 - \Lambda(M + P/\lambda, z, \overline{\sigma}_y)) = 0\).

The limiting results in Proposition 4 provide important insights into the relationship between the precision of the private and the public signals and multiple equilibrium prices in our model. When \(\sigma_y\) approaches to zero, price becomes fully revealing and the information effect dominates the substitution effect, resulting in multiple equilibrium prices. When \(\sigma_y\) approaches \(\infty\), price is extremely noisy and the information effect vanishes, resulting in a unique equilibrium price.

When the noise in the private signal, \(\sigma_s\), approaches to \(\infty\), the distribution of informed investors’ signals become dispersed and coordination among informed investors becomes difficult. Therefore, the coordination component of the information effect vanishes leading to a domination of the substitution effect and a unique equilibrium price. When \(\sigma_s\) approaches to zero, however, the multiplicity may or may not occur depending on the liquidity in the market. In a liquid market, i.e. when \(\sigma_y\) is large, price is less informative about informed investors’ demand and hence coordination even when private signals are very precise, leading to a unique equilibrium price. In an illiquid market, price could be quite informative of informed investor’s coordination especially when the distribution of their signals is concentrated. This could result in multiple equilibrium prices.

These limiting results on the precision of the private signals differ from those in the existing literature. When private signals are noisy enough, we find the equilibrium price is

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21 There exists a unique \(\overline{\sigma}_y > 0\) that satisfies \(\lambda - \alpha(1 - \Lambda(M + P/\lambda, z, \overline{\sigma}_y)) = 0\). To see this recall that by Proposition 3, \(\Lambda(M + P/\lambda, z, \sigma_y)\) is increasing in \(\sigma_y\) and goes to zero as \(\sigma_y\) goes to zero and one as \(\sigma_y\) goes to \(\infty\).
Figure 2: **Equilibrium Prices.** The dotted line, the dashed line, and the solid line in the graph each represents the aggregate demand, \( X(P, \theta) + L(P) \), when \( \sigma_s \) is 20, 60, and 200, respectively. The parameters are chosen to illustrate a stylized example where \( \sigma_y = 4 \), \( \alpha = 2 \), \( z = 20 \), \( \lambda = 1 \), \( M = 1 \), and \( \theta = 0 \).

unique. Angeletos and Werning (2005) find a similar result, but Morris and Shin (2003) find multiplicity in this limit.\(^{22}\) When private signals are extremely precise, we find multiplicity arises in an illiquid market but not in a liquid market. This result is in contrast to both Morris and Shin (2003) and Angeletos and Werning (2005). The former finds a unique equilibrium and the latter finds price multiplicity regardless of the level of market liquidity. These comparisons are also illustrated in Figure 3.

Furthermore, we identify a different channel through which multiplicity arises in asset markets. In a pure coordination problem, Morris and Shin (2003) find that, when private signals are too noisy, agents coordinate on the more informative exogenous public signal, resulting in multiplicity. Angeletos and Werning (2005), on the other hand, show that if the public signal aggregates the private signals, the opposite would happen - as private signals become more precise, so does the public signal which leads to multiplicity. By embedding the coordination problem into an asset market setting, we find that the channel that leads to multiplicity is not the informed investors' inference about the fundamental component but rather about the coordination component. To highlight the difference, note that even with close-to fully revealing private signals, in a very liquid market, it is difficult for an individual

\(^{22}\)When comparing our multiplicity vs. uniqueness results with those of Morris and Shin (2003), one must keep in mind that our results refer to price multiplicity and theirs to multiplicity in equilibrium strategies.
Figure 3: Regions of Multiplicity and Uniqueness. The $\sigma_x$ measures the exogenous noise in the private information. The $\sigma_y$ measures the exogenous noise trading. The lines in the left, middle, and right panels delineate the boundary between the multiple equilibria region and the unique equilibrium region in Morris and Shin (2003), in Angeletos and Werning (2005), and in this paper respectively.

informed investor to infer about informed investors’ overall demand for the asset at a given price. Therefore, coordination is difficult in a liquid market. This is reflected in the fact that the coordination component of the price information effect vanishes as liquidity increases. Sharp inferences about informed investors’ demand at a given price are only possible in illiquid markets. Thus, multiplicity can arise only in illiquid markets in our model.

Previously, the literature has pointed out that more informative prices may facilitate coordination and play a destabilizing role. Our results indicate that asset prices have a limited role in aggregating private information, especially in a liquid market. In fact, when we conduct numerical analysis with “reasonable” parameter values as in Gennotte and Leland (1990) and in Yuan (2005), multiple equilibria do not appear unless the market is extremely illiquid (that is, ceteris paribus, $\sigma_y$ is lower than 0.02,) or the feedback of informed investors’ investment exceeds 700.\footnote{Following Gennotte and Leland (1990) and Yuan (2005), the “reasonable” parameters are chosen so that the risky asset can be interpreted as the stock market portfolio with an expected return of 6% and a standard deviation about 20%. For the informed investors’ private signal, we experiment with the signal-to-noise ratio from 20 (as in Gennotte and Leland (1990)) to 30 (as in Yuan (2005)). The ratio of uninformed to informed investors is 5.67 to 1 to reflect the 15% overall institutional holdings of the stock market. This restriction, together with the volatility of the dividend process (which ranges from 20% to 80%), pins down the relationship between $z$ and $\lambda$. The liquidity demand, $\sigma_y$, ranges from 20 (as in Yuan (2005)) to 0.02 (as in Gennotte and Leland (1990)), and the aggregate asset supply varies between 1 and 2. The numerical examples are available upon request.}
It is important to emphasize that the source of multiplicity in this setup is different from existing noisy REE models. Yuan (2004; 2005) has shown that the information effect is always dominated by the substitution effect in a standard Grossman-Stigliz setup (1980), which implies that the equilibrium is unique. However, if there exists an additional source of uncertainty such as borrowing or short-sales constraints (as in Yuan (2005) or in Barlevy and Veronesi (2003) together with distributions with large extreme tails), or the programming trading status(es) (as in Gennotte and Leland (1990)), the information effect may dominate the substitution effect which leads to multiple equilibria. In other words, in the existing Grossman-Stigliz models (1980), the interaction between uninformed and informed investors may give rise to multiple equilibrium prices only if there is an additional source of uncertainty. In the current setup, multiplicity arises due to the interaction among heterogeneously informed investors.

We next analyze some asset pricing implications of this additional source of uncertainty. We start by examining the sensitivities of price to fundamental shocks, $\frac{\partial P}{\partial \theta}$, and external noise demand shocks, $\frac{\partial P}{\partial y}$ in the intermediate region.\footnote{Outside the intermediate region the sensitivity of price to fundamental shocks is zero, and to non-fundamental shocks is positive and constant.}

**Lemma 2** In the intermediate region, where $g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s$, the sensitivities of price to fundamental shocks and external noise demand shocks are given respectively by:

$$\frac{\partial P}{\partial \theta} = \frac{\lambda z}{2 \sigma_s} \frac{\sigma_y}{1 + \frac{\lambda z}{2 \sigma_s} \frac{\partial g(P)}{\partial P}} \quad \text{and} \quad \frac{\partial P}{\partial y} = \frac{\lambda \sigma_y}{1 + \frac{\lambda z}{2 \sigma_s} \frac{\partial g(P)}{\partial P}}.$$ 

Both sensitivities can be negative even when there is a unique equilibrium price.

As the previous lemma indicates, when there is strategic uncertainty, prices may not be monotone in the level of fundamental or noise demand shocks. Loosely speaking, this indicates that we may expect more volatility when there is strategic uncertainty in the market. To investigate this formally, we use absolute values of these sensitivities as proxies for fundamental and non-fundamental volatilities, and generate comparative static results with respect to small changes in the noise in private signals ($\sigma_s$), the volatility of noise demand ($\sigma_y$), and the coordination incentive for informed investors ($\alpha$). This analysis allows us to generate empirical testable implications regarding volatilities and strategic uncertainty.

In the next proposition we provide comparative statics results for fundamental volatility of price.
Proposition 5  
(i) As private information becomes more precise, (a) fundamental volatility of price increases, (i.e., \( \frac{\partial}{\partial \sigma} \frac{\partial P}{\partial \theta} < 0 \)) and (b) non-fundamental volatility of price increases if and only if market is illiquid enough (i.e. \( \frac{\partial}{\partial \sigma_y} \frac{\partial P}{\partial y} < 0 \Leftrightarrow \sigma_y \leq \overline{\sigma}_y \) where \( \overline{\sigma}_y \) satisfies \( \lambda - \alpha (1 - \Lambda (M + P/\lambda, z, \sigma_y)) = 0 \)).

(ii) As the volatility of noise demand decreases, (a) fundamental volatility of price increases (i.e. \( \frac{\partial}{\partial \sigma_y} \frac{\partial P}{\partial \theta} < 0 \)) and (b) change in non-fundamental volatility of price is ambiguous, but as the market becomes extremely illiquid non-fundamental volatility of price increases (i.e., for \( \sigma_y \) close to zero, \( \frac{\partial}{\partial \sigma_y} \frac{\partial P}{\partial y} < 0 \)).

(iii) As the coordination incentive for informed investors increases, both the fundamental and non-fundamental volatility increases (i.e. \( \frac{\partial}{\partial \alpha} \frac{\partial P}{\partial \theta} < 0 \) and \( \frac{\partial}{\partial \alpha} \frac{\partial P}{\partial y} < 0 \)).

Parts (i) and (ii) may appear counter-intuitive since more information should in general reduce the volatility to external noise shocks. However, a more precise private signal (or a less volatile noise demand) in this setting not only is more informative about the fundamental, but also leads to easier coordination. Part (iii) of Proposition 5 also reflects the same intuition, as coordination incentives increase so does the volatility.\(^{25}\)

5. Asset Prices as Public Signals for All Investors

In this section, we consider the case where uninformed investors also condition their demand on the price of the risky asset. Thus, asset prices not only coordinate informed investors’ beliefs but also transmit information to uninformed investors. In this section we keep all the assumptions on the information structure of informed investors the same, but to make uninformed investors inference problem well defined, we make an additional assumption that they all observe \( [\hat{\theta}, \overline{\theta}] \) which is the support from which \( \theta \) is uniformly drawn. The following definition describes the corresponding equilibrium concept.

Definition 3  An equilibrium is a price function, \( P(\theta, y) \), strategies, \( \pi(s_i, P) : [\hat{\theta} - \sigma_s, \overline{\theta} + \sigma_s] \times \mathbb{R} \rightarrow \{0, 1\} \), for each informed investor where \( \pi_i(s_i, P) = 1 \) if the informed investor buys the stock and 0 otherwise, and the corresponding aggregate demands, such that:

\(^{25}\)The comparative static results in Proposition 5 (with the exception of part (ii-b)) apply both when the equilibrium price is unique and when it is not. So these results are not just due to multiplicity.
For informed agent $i$, $\pi(s_i,P) \in \arg\max_u E[U(\pi, X(P,\theta), \theta, P)|\mathcal{F}_i]$, where $\mathcal{F}_i = \{s_i, P\}$ is informed investor $i$’s information set. The aggregate demand from informed agents, $X(P,\theta)$, is given by $\frac{1}{\sigma_s} \int_s \pi(s,P) h(\frac{s-\theta}{\sigma_s}) ds$.

Uninformed long-term investors demand, $L(P)$, is given by $w(E(V+\nu|\mathcal{F}_{ui})-P)/\rho V_{ar}(V+\nu|\mathcal{F}_{ui})$ where $\mathcal{F}_{ui} = \{P\}$ is uninformed investors’ information set.

The market clearing condition is satisfied: $X(P,\theta) + L(P) + \sigma_y y = M$.

Before we present the equilibrium solution(s), we first make a note of a technical condition close to the upper and lower bound of $\theta$. Suppose that informed investors are following a cutoff strategy. At a given price, consider the payoff of an informed investor assuming that her signal is the cutoff signal. As her signal increases the agent will believe that the asset on average has a higher fundamental. However, close to the boundaries an additional countervailing effect appears. For example, as the distance between this signal and the upper boundary, $\theta$, falls below $\sigma_s$, the informed agent believes that fewer informed investors buy the stock. In fact, if the signal is close to the upper boundary, $\theta$, then this agent will believe that no matter what the true fundamental is less than half of the informed investors will buy the asset. Therefore, close to the boundaries, the payoff from buying the asset may in fact decrease as the signal increase, which may lead to equilibrium multiplicity. Since this is a technical problem that appears only close to the boundaries we assume that in a small neighborhood of $\theta$ the informed investors receive an arbitrarily negative (positive) private payoff. The following proposition provides a characterization of monotone cutoff equilibrium as this neighborhood vanishes, and the informed and uninformed investors face the same dividend function in the limit.

**Proposition 6** Suppose that for the uninformed investors dividend function is given by $f(X,\theta)$, and for the informed investors it is given by $f_\epsilon(X,\theta)$ which is equal to $f(X,\theta)$ if $\theta + \epsilon \leq \theta \leq \bar{\theta} - \epsilon$, $-\infty$ if $\theta < \theta + \epsilon$ and $\infty$ if $\theta > \bar{\theta} - \epsilon$. Consider the game of incomplete information described in this section and its equilibrium as described in Definition 3. In this game,

- as $\epsilon$ approaches zero, there exists a unique monotone equilibrium, that is, there is a unique function $g : \mathbb{R} \rightarrow [\theta + \sigma_s, \bar{\theta} - \sigma_s]$ such that the equilibrium strategies of informed investors are given by $\pi(s,P) = 1$ if $s \geq g(P)$ and 0 otherwise.
informed investors’ aggregate demand in this monotone equilibrium, \( X(P, \theta) \), is uniquely characterized by Eq. (6). For \( \sigma_v \) large enough, uninformed investors demand, \( L(P) \), is uniquely characterized by

\[
L(P) = w \frac{E[V + \nu|P] - P}{\rho \text{Var}[V + \nu|P]};
\]  

(16)

and the equilibrium price \( P(\theta, y) \) is an element of the set of \( P \) that satisfies

\[
P = \begin{cases} 
\lambda \sigma_y y - \lambda \hat{M}(P) & \text{if } \frac{g(P) - \theta}{\sigma_s} > 1 \\
\frac{\lambda z}{2} \left( 1 - \frac{g(P) - \theta}{\sigma_s} \right) + \lambda \sigma_y y - \lambda \hat{M}(P) & \text{if } -1 \leq \frac{g(P) - \theta}{\sigma_s} \leq 1 \\
\lambda z + \lambda \sigma_y y - \lambda \hat{M}(P) & \text{if } \frac{g(P) - \theta}{\sigma_s} < -1 
\end{cases}
\]  

(17)

where \( \hat{M}(P) = M - L(P) - P/\lambda \).

This proposition shows that, even when uninformed investors learn from price, an equilibrium in cutoff strategies exists and the informed investors’ equilibrium strategies are uniquely determined. Moreover, informed investors’ aggregate demand is characterized by the same equation as before, Eq. (6). The main difference between the two cases is that the price sensitivity of uninformed investors’ demand is no longer constant but varies with price.

The right hand side of Eq. (16) depends implicitly on \( L(P) \). We show in the Appendix that \( L(P) \) is a solution to a complicated algebraic equation which is difficult to express in closed-form, but can easily be solved numerically. Once we have a solution for \( L(P) \), Proposition 6 shows a simple procedure to solve for equilibrium prices. Specifically, using \( L(P) \), we can first compute \( \hat{M}(P) \). Next, we consider a fictitious economy where uninformed investors do not learn from prices, and the asset supply is given by \( \hat{M}(P) \) and solve for the equilibrium strategy, \( g(P) \), of informed investors. Finally, given \( \hat{M}(P) \) and \( g(P) \), we solve Eq. (17) to find the market clearing prices.

Now we illustrate this procedure in the linear example where \( f(X, \theta) = \alpha X + \theta \). The next lemma characterizes the equilibrium strategies of informed investors.

**Lemma 3** When the dividend payoff function is \( f(X, \theta) = \alpha X + \theta \), the equilibrium cutoff strategy, \( g(P) \), is unique and as \( \epsilon \) goes to zero it is characterized by,

\[
g(P) = P + \sigma_s - \left( \alpha + \frac{2\sigma_s}{z} \right) \left( \hat{M}(P) + \frac{P}{\lambda} - E \left( \sigma_y y \bigg| \hat{M}(P) + \frac{P}{\lambda} - z \leq \sigma_y y \leq \hat{M}(P) + \frac{P}{\lambda} \right) \right)
\]  

(18)

when \( \theta + \sigma_s \leq g(P) \leq \bar{\theta} - \sigma_s \).

If \( P \) is such that \( g(P) \) computed from Eq. (18) is more (less) than \( \bar{\theta} - \sigma_s (\theta + \sigma_s) \), then \( g(P) \) is set to
Figure 4: **Equilibrium Strategies and Demands: Backward-bending Uninformed and Aggregate Demands.** The dotted line, the dashed line, and the solid line in the graph each represents the uninformed investor demand, \(L(P)\), the informed investor demand, \(X(P, \theta)\), the aggregate demand, \(X(P, \theta) + L(P)\) The parameters are chosen to illustrate a stylized example where \(\sigma_y = 4\), \(\alpha = 2\), \(z = 20\), \(M = 1\), \(\theta = 0\), \(\sigma_s = 60\), \(w = 1.9\), \(\sigma_v = 1\), \(\rho = 1\), \(\hat{\theta} = -100\), and \(\bar{\theta} = 100\).

Note that the equilibrium strategies are the same as those in Eq. (9) in Lemma 1 with \(M\) replaced by \(\hat{M}(P)\). This is because the asset supply in the fictitious economy is \(\hat{M}(P)\).

When uninformed investors make inferences based on price there is an additional source for multiplicity. The following equation describes the price sensitivity of aggregate demand in the intermediate region where \(\theta \in [g(P) - \sigma_s, g(P) + \sigma_s]\) and illustrates the additional information effect from uninformed investors:

\[
\frac{\partial (X(P, \theta) + L(P))}{\partial P} = -\left(\frac{1}{\lambda}\right)^{\text{Substitution Effect}} + \left(\frac{z}{2\sigma_s}\right)^{\text{Information Effect (from Informed)}} + \left(\frac{z \alpha}{2\sigma_s} \left(1 - \frac{1}{\lambda}\right) \left(1 - \frac{1}{\lambda}\right)^{\text{Coordination Component}} + \left(1 - \frac{1}{\lambda}\right)^{\text{Fundamental Component}}\right)^{\text{Information Effect (from Informed)}}
\]

\[
\bar{\theta} - \sigma_s (\bar{\theta} + \sigma_s).
\]
To illustrate the importance of this additional effect, we provide a numerical example in Figure 4. In this example, informed investors, as an aggregate, do not treat the asset as a Giffen good, yet through the uninformed investors’ information effect aggregate demand has a backward-bending region. The following corollary is immediate following this numerical example.

**Corollary 2** When the dividend payoff function is \( f(X, \theta) = \alpha X + \theta \), the aggregate uninformed investor demand may have a backward-bending region even if the aggregate informed investor demand is downward sloping.

Corollary 2 reconfirms the finding in the existing REE literature (Gennotte and Leland (1990); Barlevy and Veronesi (2003); Yuan (2005)) as it identifies another channel for multiplicity or excess volatility in the market: the nonlinear inference from uninformed investors when the market has an additional source of uncertainty. In this setting, the additional source of uncertainty is strategic uncertainty.

6. Conclusion

In this paper, we present a REE framework to analyze strategic uncertainty by embedding the coordination game within the asset market through strategic complementarity in investment. In presence of strategic uncertainty, asset price is not only informative of the fundamentals but also of the likelihood of coordination among informed investors. By characterizing the coordination and the fundamental components of the information effect of asset price in relation to its traditional substitution effect, we highlight sources of volatility in asset markets and identify testable asset pricing implications.

More importantly, we find that asset prices, as an endogenous public signal, play a limited role in aggregating diverse private information and in coordinating investors’ actions. Specifically, even though the precision of the endogenous public signal, that is, asset prices, increases with the precision of private information, multiplicity may not occur for very small private noises if the asset market is liquid. This result extends the understanding of the role of prices in coordination games. In the existing literature prices have been regarded as destabilizing since they facilitate coordination by allowing informed investors to make inferences. We find, however, that the source of volatility and multiplicity is not the informed investors’ inference about the fundamental component but rather about the coordination.
component. Even if private signals are close-to fully revealing of the fundamentals, in a liquid market, it is difficult for an individual informed investors to form beliefs regarding other informed investors’ actions. Therefore, coordination on prices are difficult. On the other hand, if the asset market is illiquid, multiplicity in equilibrium prices may occur when private information becomes extremely precise. However, this is because it is easier to forecast aggregate informed investors’ demand at a given price, rather than because the prices are fully-revealing.

Moreover, we explicitly characterize uninformed investors’ behavior and allow them to make inferences based on price just as informed investors. The non-linear inference of uninformed investors introduces an additional channel of volatility in the asset market in the presence of strategic uncertainty.

References


Appendix A

Proof of Proposition 1
Suppose that all informed investors follow a cutoff strategy with cutoff $\kappa$. In this case, informed investors’ aggregate demand is given by:

$$X(\theta) = \begin{cases} 
0 & \text{if } \frac{\kappa - \theta}{\sigma_s} > 1 \\
\frac{z}{2} \left(1 - \frac{\kappa - \theta}{\sigma_s}\right) & \text{if } -1 \leq \frac{\kappa - \theta}{\sigma_s} \leq 1 \\
z & \text{if } \frac{\kappa - \theta}{\sigma_s} < -1 
\end{cases}$$

Since an informed investor who receives the cutoff signal is indifferent between buying the asset or not, to solve for the cutoff value we need to solve for the values of $\kappa$ that satisfy:

$$\int_{\kappa-\sigma_s}^{\kappa+\sigma_s} f(X(\theta), \theta) \, d\theta - \lambda(X(\theta) - M)$$

$$= \int_{\kappa-\sigma_s}^{\kappa+\sigma_s} f\left(\frac{z}{2} \left(1 - \frac{\kappa - \theta}{\sigma_s}\right), \theta\right) \, d\theta - \lambda\left(\frac{z}{2} \left(1 - \frac{\kappa - \theta}{\sigma_s}\right) - M\right) = 0.$$

Using the change of variables $w = (\theta - \kappa)/\sigma_s$ we can rewrite the above equation as:

$$\int_{-1}^{1} f\left(\frac{z}{2} (1 + w), \sigma_s w + \kappa\right) \, dw - \lambda\left(\frac{z}{2} (1 + w) - M\right) = 0. \quad (20)$$

Since $f_\theta > 0$, the left hand side of the above expression is strictly increasing in $\kappa$ so there is a unique value of $\kappa$ that satisfies the equality. To finish the proof we need to show that an informed investor who receives a signal less that $\kappa$ strictly prefers not to buy the asset, and more than $\kappa$ strictly prefers to buy the asset. To demonstrate this, let’s start with the case where an informed investor receives signal $s < \kappa - \sigma_s$. In this case:

$$\int_{\kappa-\sigma_s}^{\kappa+\sigma_s} f(0, \theta) \, d\theta + \lambda M = \int_{-1}^{1} f(0, \sigma_s w + \kappa) \, dw + \lambda M$$

$$< \int_{-1}^{1} f\left(\frac{z}{2} (1 + w), \sigma_s w + \kappa\right) \, dw - \lambda\left(\frac{z}{2} (1 + w) - M\right) = 0.$$

The inequality above follows since $f_X > \lambda$. So in this case the informed investor strictly prefers not to buy the asset. Using similar arguments we can show that an informed investor who receives signal $\kappa - \sigma_s \leq s < \kappa$ also strictly prefers not to buy the asset, and an informed investor who receives signal $s > \kappa$ strictly prefers to buy the asset.

Proof of Corollary 1
The proof follows by substituting $M = 0$ and $f(X, \theta) = \alpha X + \theta$ in Eq. 20 and solving for $\kappa$.

Proof of Proposition 2
For a given private signal, $s, \theta$ is distributed uniformly $[s - \sigma_s, s + \sigma_s]$. The following equation
expresses the informed investors’ conditional expectation of the risky asset’s dividend payoff.

\[
\int_{-\infty}^{\infty} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta = \int_{-\infty}^{g(P) - \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta + \int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta + \int_{g(P) + \sigma_s}^{\infty} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta
\]

(21)

where \( h(\theta \mid s, P) \) is the density of \( \theta \) conditional on \( s \) and \( P \).

First we consider the first term on the right hand side of equation (21). Using Bayes rule this term can be written as:

\[
\int_{-\infty}^{g(P) - \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta = \text{prob}(\theta < g(P) - \sigma_s \mid s, P) \int_{-\infty}^{g(P) - \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P, \theta < g(P) - \sigma_s) \, d\theta.
\]

Note that

\[
\text{prob}(\theta < g(P) - \sigma_s \mid s, P) = \frac{(\min\{s + \sigma_s, g(P) - \sigma_s\} - (s - \sigma_s))}{2\sigma_s}.
\]

Moreover, by Eq. (7) price is uninformative about \( \theta \) conditional on \( \theta < g(P) - \sigma_s \). Thus, the posterior is uniform over this range and

\[
h(\theta \mid s, P, \theta < g(P) - \sigma_s) = \frac{1}{\min\{s + \sigma_s, g(P) - \sigma_s\} - (s - \sigma_s)}.
\]

Combining we obtain,

\[
\int_{-\infty}^{g(P) - \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta = \frac{1}{2\sigma_s} \int_{s - \sigma_s}^{\min\{s + \sigma_s, g(P) - \sigma_s\}} f(X(P, \theta), \theta) \, d\theta.
\]

Similarly the second term on the right hand side of equation (21) can be written as:

\[
\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P) \, d\theta = \text{prob}(g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s)
\]

\[
\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta \mid s, P, g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s) \, d\theta.
\]

Let

\[
t \equiv \left(\frac{2\sigma_s}{\lambda z} P + \frac{2\sigma_s}{z} M + g(P) - \sigma_s\right).
\]

Note that in this region \( t = \theta + (2\sigma_s \sigma_y/z)g \). Hence, \( t \) is a sufficient statistic for the information
Finally, we can write the third term on the right hand side of equation (21) as:

\[
\begin{align*}
&\frac{h(\theta|s, P, g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s)}{f_0 - f_{s-s, s+\sigma_s} \cap [g(P) - \sigma_s, g(P) + \sigma_s]} \\
&\text{if } \theta \in [s - \sigma_s, s + \sigma_s] \cap [g(P) - \sigma_s, g(P) + \sigma_s] \\
&0 \quad \text{otherwise}
\end{align*}
\]

Thus,

\[
\int_{g(P)-\sigma_s}^{g(P)+\sigma_s} f(X(\theta), \theta) h(\theta|s, P) d\theta = \frac{||[s - \sigma_s, s + \sigma_s] \cap [g(P) - \sigma_s, g(P) + \sigma_s]||}{2\sigma_s}
\]

where \( ||[s - \sigma_s, s + \sigma_s] \cap [g(P) - \sigma_s, g(P) + \sigma_s]|| \) is the length of the interval.

Finally, we can write the third term on the right hand side of equation (21) as:

\[
\int_{g(P)-\sigma_s}^{g(P)+\sigma_s} f(X(\theta), \theta) h(\theta|s, P) d\theta = \frac{1}{2\sigma_s} \int_{\max\{s-\sigma_s, g(P)+\sigma_s\}}^{s+\sigma_s} f(X(\theta), \theta) d\theta.
\]

To solve for the cutoff strategy, we consider the agent who receives the cutoff signal, \( s = g(P) \).

For this agent the first and the third terms are 0 so the indifference condition becomes:

\[
\int_{g(P)-\sigma_s}^{g(P)+\sigma_s} f\left(\frac{z}{2} \left(1 - \frac{g(P) - \theta}{\sigma_s}\right), \theta\right) \phi\left(\frac{\theta - t}{2\sigma_s\lambda z}ight) d\theta = P
\]

To find the cutoff value(s) \( g(P) \) for a given \( P \), we need to find those values of \( \kappa \) that satisfy:

\[
\int_{\kappa-\sigma_s}^{\kappa+\sigma_s} f\left(\frac{z}{2} \left(1 - \frac{\kappa - \theta}{\sigma_s}\right), \theta\right) \phi\left(\frac{\theta - \kappa - 2g(P)/(\lambda z)-2\sigma_s M/z+\sigma_s}{2\sigma_s\lambda z}ight) d\theta = P
\]

Next we show that there is a unique \( \kappa \) that satisfies the above equation. Using change of variables \( x = \frac{\theta - \kappa}{\sigma_s} \) we can rewrite it as:

\[
\int_{-1}^{1} f\left(\frac{z}{2} (1 + x), \kappa + x\sigma_s\right) \phi\left(\frac{x - 2P/(\lambda z) - 2M/z + 1}{2\sigma_s\lambda z}\right) dx = P
\]

Since \( f(X, \theta) \) is increasing in \( \theta \), it is also increasing in \( \kappa \) for a given \( P \). So the left-hand side is

\[
31
\]
increasing in \( \kappa \) if and only if
\[
\int_{-1}^{1} \phi \left( \frac{x - 2P/(\lambda z) - 2M/z + 1}{2\sigma_y/z} \right) dx > 0
\]
which is clearly true.

We have just shown that for a given \( P \) there is a unique signal, \( \kappa \), that makes an informed investor indifferent between acquiring the asset or not. Moreover, again since \( f(X, \theta) \) is increasing in \( \theta \), for a given \( P \) the payoff to acquiring the asset is strictly positive if \( s > \kappa \) and strictly negative if \( s < \kappa \). Therefore, an informed investor buys if and only if her private signal exceeds \( g(P) = \kappa \), which completes the proof of the first bullet point in Proposition 2. The second and third bullet points follow immediately from the first one.

**Proof of Lemma 1**

Substituting \( f(X, \theta) = \alpha X + \theta \) in the indifference condition in Eq. (22) and rearranging we obtain:
\[
g(P) = \kappa = P - \frac{\alpha z}{2} - \frac{\int_{-1}^{1} \left( \frac{\alpha z}{2} + \sigma_s \right) x \phi \left( \frac{x - 2P/(\lambda z) - 2M/z + 1}{2\sigma_y/z} \right) dx}{\int_{-1}^{1} \phi \left( \frac{x - 2P/(\lambda z) - 2M/z + 1}{2\sigma_y/z} \right) dx}.
\]

To finish the proof we use a change of variables \( u = (2P/(\lambda z) + 2M/z - 1 - x)/(2\sigma_y/z) \), and rearrange one more time to obtain:
\[
g(P) = P - \frac{\alpha z}{2} - \frac{\int_{\frac{P}{\lambda' + M}}^{\frac{P}{\lambda' + M} - \frac{2P}{\sigma_y}} \left( \frac{\alpha z}{2} + \sigma_s \right) (2P/(\lambda z) + 2M/z - 1 - 2\sigma_y u/z) \phi(u) du}{\int_{\frac{P}{\lambda' + M} - \frac{2P}{\sigma_y}}^{\frac{P}{\lambda' + M}} \phi(u) du}.
\]

**Proof of Proposition 3**

To obtain the necessary and sufficient condition for \( g(P) \) to be increasing in \( P \), we first prove \( \mathbb{E} \left[ u \left| \frac{P/\lambda + M - z}{\sigma_y} \leq u \leq \frac{P/\lambda + M}{\sigma_y} \right. \right] \) is increasing in \( P \). The following two lemmas are useful in this proof as well as the later sensitivity analysis. First of these two lemmas generalizes a result from Burdett (1996) from right-truncated distributions to doubly truncated distributions and is used to prove the second one. The second lemma, among other things, gives us immediately that the above conditional expectation is increasing in \( P \).

**Lemma 4** Suppose that \( f \) is a log-concave and differentiable density function on \( \mathbb{R} \) and \( h < k \). Then the truncated variance \( \text{Var}(u|h \leq u \leq k) \) computed using the density \( f \) is decreasing in \( h \) and increasing in \( k \).
Proof. From Proposition 4 in Burdett (1996), it follows immediately that variance is increasing in $k$ for a fixed $h$. In applying the proposition we only need to notice that for a left truncated distribution we can replace negative infinity with $h$ everywhere and the argument goes through exactly. The fact that for a fixed $k$ variance is decreasing in $h$ then follows immediately from applying this result to the density $\tilde{f}(x) = f(-x)$. ■

Lemma 5 Let $E\left(u \mid \frac{a+x}{b} \geq u \geq \frac{x}{b}\right)$ be the truncated expectation of the standard normal distribution with $a, b > 0$ and let $\Lambda(a + x, a, b) \equiv b \frac{\partial E(u \mid \frac{a+x}{b} \geq u \geq \frac{x}{b})}{\partial x}$. Then the following statements are true.

(i) $E\left(u \mid \frac{a+x}{b} \geq u \geq \frac{x}{b}\right)$ is strictly increasing in $x$.

(ii) $\Lambda(a + x, a, b)$ is no greater than one and bounded away from zero, and when $x \leq 0 \leq a + x$, $\Lambda(a + x, a, b)$ is increasing in $b$.

(iii) $\Lambda(a + x, a, b)$ goes to one as $b \to \infty$. When $x < 0 < a + x$, $\Lambda(a + x, a, b)$ goes to zero as $b \to 0$.

Proof. Taking the derivative of $E\left(u \mid \frac{a+x}{b} \geq u \geq \frac{x}{b}\right)$ with respect to $x$ and multiplying by $b$ we obtain

$$ \Lambda(a + x, a, b) = \frac{\left[\frac{a+x}{b} \phi\left(\frac{a+x}{b}\right) - \frac{x}{b} \phi\left(\frac{x}{b}\right)\right] \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du - \left[\phi\left(\frac{a+x}{b}\right) - \phi\left(\frac{x}{b}\right)\right] \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} u \phi(u) \, du}{2 \left(\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du \right)^2} $$

$$ = \frac{\phi\left(\frac{a+x}{b}\right) \left[\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du - \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} (u - \frac{x}{b}) \phi(u) \, du\right] + \phi\left(\frac{x}{b}\right) \left[\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} (u - \frac{x}{b}) \phi(u) \, du\right]}{2 \left(\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du \right)^2}.$$  

Note that $\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} (u - \frac{x}{b}) \phi(u) \, du$ is strictly positive and $\frac{x}{b} \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du > \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} (u - \frac{x}{b}) \phi(u) \, dz$. Therefore, the denominator is strictly positive. This proves that $\Lambda(a + x, a, b)$ is strictly positive. Since $b > 0$, this implies that $E\left(u \mid \frac{a+x}{b} \geq u \geq \frac{x}{b}\right)$ is strictly increasing in $x$, proving part (i).

Next, we prove part (ii). First note that using integration by part, the following is true.

$$\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du = \frac{a+x}{b} \phi\left(\frac{a+x}{b}\right) - \frac{x}{b} \phi\left(\frac{x}{b}\right) + \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} u \phi(u) \, du.$$  

Using this, as well as the fact that

$$\left(\phi\left(\frac{x}{b}\right) - \phi\left(\frac{a+x}{b}\right)\right) = \int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} u \phi(u) \, du,$$

we simplify the expression of $\Lambda(a + x, a, b)$ as

$$\Lambda(a + x, a, b) = 1 - \frac{\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} u^2 \phi(u) \, du}{\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du} - \left(\frac{\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} u \phi(u) \, du}{\int\limits_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du}\right)^2 = 1 - \text{Var}\left(u \mid \frac{x}{b} \leq u \leq \frac{a+x}{b}\right).$$
It immediately follows that $\Lambda(a+x,a,b)$ is no greater than one. Moreover by lemma 4, when $x \leq 0 \leq a+x$, truncated variance $Var\left(u|\bar{z} \leq u \leq \frac{a+x}{b}\right)$ is decreasing in $b$, thus $\frac{\partial\Lambda(a+x,a,b)}{\partial b} \geq 0$. This proves part (ii).

Next, we show the limiting results in part (iii). To identify lower and upper bounds of $\Lambda(a+x,a,b)$ first note that,

$$\frac{\frac{a}{b}\phi\left(\frac{x}{b}\right)}{\int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du} = \left[\left(\frac{a+x}{b}\right)\phi\left(\frac{a+x}{b}\right) - \frac{x}{b}\phi\left(\frac{x}{b}\right)\right] \int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du - \left[\phi\left(\frac{a+x}{b}\right) - \phi\left(\frac{x}{b}\right)\right] \int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi\left(\frac{x}{b}\right) \, du$$

and

$$\frac{\frac{a}{b}\phi\left(\frac{a+x}{b}\right)}{\int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du} = \left[\left(\frac{a+x}{b}\right)\phi\left(\frac{a+x}{b}\right) - \frac{x}{b}\phi\left(\frac{x}{b}\right)\right] \int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du - \left[\phi\left(\frac{a+x}{b}\right) - \phi\left(\frac{x}{b}\right)\right] \int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi\left(\frac{x}{b}\right) \, du$$

When $\phi\left(\frac{a+x}{b}\right) > \phi\left(\frac{x}{b}\right)$,

$$\frac{\frac{a}{b}\phi\left(\frac{x}{b}\right)}{\int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du} \leq \Lambda(a+x,a,b) \leq \frac{\frac{a}{b}\phi\left(\frac{a+x}{b}\right)}{\int_{\frac{x}{b}}^{\frac{a+x}{b}} \phi(u) \, du}.$$

When $\phi\left(\frac{x}{b}\right) > \phi\left(\frac{a+x}{b}\right)$, the inequalities above are reversed.

In either case applying L’Hopital’s rule, we show both the lower and the upper bounds of $\Lambda(a+x,a,b)$ approach to one as $b \to \infty$.

Next we show the limiting result as $b$ goes to zero. When $x < 0 < a+x$, the truncated variance $Var\left(u|\bar{z} \leq u \leq \frac{a+x}{b}\right)$ approaches to one and $\Lambda(a+x,a,b)$ approaches to 0. This concludes the proof. ■

From part (i) of the previous lemma, we immediately conclude that $E\left[u|\frac{P/L+M-z}{\sigma_u} \leq u \leq \frac{P/L+M}{\sigma_u}\right]$ is increasing in $P$. Moreover, its derivative with respect to $P$ is given by $(1/\lambda\sigma_y)\Lambda(M+P/\lambda, z, \sigma_y)$. From this fact and Eq. (23) we find:

$$\frac{\partial g(P)}{\partial P} = 1 - \frac{\alpha z + 2\sigma_s}{\lambda z} \left(1 - \Lambda(M+P/\lambda, z, \sigma_y)\right).$$

Therefore, $g(P)$ is increasing at a given $P$ if and only if $(\alpha + 2\sigma_s/\lambda)(1 - \Lambda(M+P/\lambda, z, \sigma_y)) < 1$. This proves the first statement in Proposition 3. The limiting results in the proposition follow straightforwardly from parts (ii) and (iii) in Lemma 5.

**Proof of Proposition 4**

From Eq. (6) and (23), we learn that the slope of informed investors’ aggregate demand function
in the intermediate region of \(-1 \leq (g(P) - \theta)/\sigma_s \leq 1\) is

\[
\frac{\partial X(P, \theta)}{\partial P} = \left(-\frac{z}{2\sigma_s}\right) \left(1 - \frac{\alpha z + 2\sigma_s}{\lambda z}(1 - \Lambda(M + P/\lambda, z, \sigma_y))\right).
\]

Since the slope of uninformed investors’ aggregate demand curve is \(-1/\lambda\), the slope of aggregate demand curve of the economy in the intermediate region is

\[
\frac{\partial(X(P, \theta) + L(P))}{\partial P} = \frac{(\alpha - \lambda)z}{2\lambda\sigma_s} - \frac{\alpha z + 2\sigma_s}{2\lambda\sigma_s}\Lambda(M + P/\lambda, z, \sigma_y).
\]

Therefore, the necessary and sufficient condition for unique equilibrium price is \((\alpha + 2\sigma_s/z)\Lambda(M + P/\lambda, z, \sigma_y) > \alpha - \lambda\).

When \(\sigma_y\) goes to infinity, that is, when the public signal is extremely noisy, \(\Lambda(M + P/\lambda, z, \sigma_y) = 1\) by Lemma 5. This implies that both the informed investors’ demand curve and the aggregate demand curve of the economy are downward sloping and the equilibrium price is unique.

When \(\sigma_y\) goes to zero, that is, when the public signal is extremely informative, in the region where \(-\lambda M < P < -\lambda(M - z)\), \(\Lambda(M + P/\lambda, z, \sigma_y) = 0\) by lemma 5. This implies that both the informed investors’ demand curve and the aggregate demand curve of the economy have backward-bending regions and there are multiple equilibrium prices.

**Proof of Proposition 5**

The sensitivity of price to fundamental and external noise demand shocks in the intermediate region can be computed from Eq. 7 as:

\[
\frac{\partial P}{\partial \theta} = \frac{\lambda z}{2\sigma_s(1 + \frac{\lambda z}{2\sigma_s} \frac{\partial g(P)}{\partial P})} = \frac{\lambda z}{2\sigma_s \Lambda(M + P/\lambda, z, \sigma_y) + \lambda z - \alpha z(1 - \Lambda(M + P/\lambda, z, \sigma_y))}
\]

and

\[
\frac{\partial P}{\partial y} = \frac{\lambda \sigma_y}{1 + \frac{\lambda z}{2\sigma_s} \frac{\partial g(P)}{\partial P}} = \frac{\lambda \sigma_y}{\frac{\lambda z - \alpha z(1 - \Lambda(M + P/\lambda, z, \sigma_y))}{2\sigma_s} + \Lambda(M + P/\lambda, z, \sigma_y)}.
\]

Parts (i-a), (iii-a) and (iii-b) follow immediately by taking the appropriate derivatives and noticing that \(0 \leq \Lambda(M + P/\lambda, z, \sigma_y) \leq 1\).

To see Part (i-b) we first compute the derivative \(\frac{\partial}{\partial \sigma_s} \frac{\partial P}{\partial y}\) and notice that it has the same sign as \(\lambda - \alpha(1 - \Lambda(M + P/\lambda, z, \sigma_y))\). Recall that \(\lambda < \alpha\). Moreover, by Proposition 3, \(\Lambda(M + P/\lambda, z, \sigma_y)\) is increasing in \(\sigma_y\) and goes to zero as \(\sigma_y\) goes to zero and one as \(\sigma_y\) goes to \(\infty\). Therefore, \(\frac{\partial}{\partial \sigma_s} \frac{\partial P}{\partial y} < 0\) if and only if \(\sigma_y < \sigma_y^*\) where \(\sigma_y^*\) satisfies \(\lambda - \alpha(1 - \Lambda(M + P/\lambda, z, \sigma_y)) = 0\).

To see Part (ii-a) we compute the derivative \(\frac{\partial}{\partial \sigma_y} \frac{\partial P}{\partial y}\) and notice that it has the same sign as \(-\partial \Lambda(M + P/\lambda, z, \sigma_y)/\partial \sigma_y\). Since \(\Lambda(M + P/\lambda, z, \sigma_y)\) is increasing in \(\sigma_y\) the result follows.
Finally to see part Part (ii-b) first we compute the derivative

\[
\frac{\partial \partial P}{\partial \sigma y \partial y} = \frac{2 \lambda \sigma_s}{\lambda z - \alpha z + (\alpha z + 2 \sigma_s) \Lambda(M + P/\lambda, z, \sigma_y)} - \frac{2 \lambda \sigma_s \sigma_y (\alpha + 2 \sigma_s)}{(\lambda z - \alpha z + (\alpha z + 2 \sigma_s) \Lambda(M + P/\lambda, z, \sigma_y))^2} \frac{\partial \Lambda(M + P/\lambda, z, \sigma_y)}{\partial \sigma y}.
\]

Note that the second term is always negative and the first term becomes negative as \(\sigma_y\) approaches zero. Thus, \(\frac{\partial \partial P}{\partial \sigma y \partial y} < 0\) for \(\sigma_y\) close to zero, completing the proof.

**Proof of Proposition 6**

Given a private signal \(s\), a price \(P\) and the corresponding equilibrium demand of uninformed investors \(L(P)\), the informed investors’ inference problem can be solved just like in the proof of Proposition 2, by replacing the asset supply \(M\) with \(\hat{M}(P)\) everywhere. The only difficulty arises when \(s\) is close to the boundaries, in particular, when \(s < \theta + \sigma_s\) or \(s > \overline{\theta} - \sigma_s\). In these cases we need to adjust the formulas appropriately.

Suppose that the informed investors dividend function is given by \(f_\epsilon(X, \theta)\). Recall that \(f_\epsilon(X, \theta) = -\infty\) if \(\theta < \theta + \epsilon\) and \(f_\epsilon(X, \theta) = \infty\) if \(\theta > \overline{\theta} - \epsilon\). Everywhere else \(f_\epsilon(X, \theta)\) is the same as \(f(X, \theta)\) and satisfies our earlier assumptions, in particular it is increasing in both arguments. As in the proof of Proposition 2, next we consider an agent who receives the cutoff signal \(s = \kappa\). Let’s denote this agent’s utility from buying the asset by \(W(\kappa)\) which can be written (after using a change variables) as,

\[
W(\kappa) = \frac{\int_U f_\epsilon\left(\frac{x}{\kappa} + \frac{x}{\kappa + \sigma_s}, \phi\left(\frac{x - 2P/\lambda z - 2M/z + 1}{2\sigma_y/z}\right)\right) \ dx}{\int_L \phi\left(\frac{x - 2P/\lambda z - 2M/z + 1}{2\sigma_y/z}\right) \ dx} \tag{24}
\]

where

\[
L = \begin{cases} 
\frac{(\theta - \kappa)/\sigma_s}{\kappa} & \text{if } \theta - \sigma_s < \kappa < \theta + \sigma_s \\
-1 & \text{if } \theta + \sigma_s < \kappa < \overline{\theta} + \sigma_s 
\end{cases} \quad \text{and} \quad U = \begin{cases} 
1 & \text{if } \theta - \sigma_s < \kappa < \overline{\theta} - \sigma_s \\
\frac{(\overline{\theta} - \kappa)/\sigma_s}{\kappa} & \text{if } \overline{\theta} - \sigma_s < \kappa < \overline{\theta} + \sigma_s 
\end{cases}
\]

It is easy to see that the expression in Eq. (24) is increasing in \(\kappa\) for \(\theta + \sigma_s + \epsilon \leq \kappa \leq \overline{\theta} - \sigma_s - \epsilon\). If \(\theta - \sigma_s \leq \kappa < \theta + \sigma_s + \epsilon\) then the expression in Eq. (24) is \(-\infty\). Similarly, if \(\overline{\theta} - \sigma_s - \epsilon < \kappa \leq \overline{\theta} + \sigma_s\) then the expression in Eq. (24) is \(\infty\).

Next, we construct \(g(P)\). Now, suppose that \(P < W(\theta + \sigma_s + \epsilon)\). Then if an agent receives the signal \(\theta + \sigma_s + \epsilon\), by construction his payoff from buying the asset is less than the price, thus she would not buy the asset. The same is true for all agents with signals that are less than \(\theta + \sigma_s + \epsilon\), but an agent with a higher signal expects to make \(\infty\) by buying the asset so she will buy the asset. Therefore, in this range cutoff is given by \(g(P) = \theta + \sigma_s + \epsilon\). By a symmetric argument
Let \( g(P) = \bar{\theta} - \sigma_s - \epsilon \) if \( P > W(\bar{\theta} - \sigma_s - \epsilon) \). Thus we can write the (unique) cutoff strategy \( g(P) \) as

\[
g(P) = \begin{cases} 
\hat{\theta} + \sigma_s + \epsilon & \text{if } P < W(\hat{\theta} + \sigma_s + \epsilon) \\
W^{-1}(P) & \text{if } W(\hat{\theta} + \sigma_s + \epsilon) \leq P \leq W(\bar{\theta} - \sigma_s - \epsilon) \\
\bar{\theta} - \sigma_s - \epsilon & \text{if } P > W(\bar{\theta} - \sigma_s - \epsilon)
\end{cases}
\]

This completes the proof of the first bullet point in the Proposition 6.

Uninformed investors’ inference problem is similar to informed investors’ except that uninformed investors’ information set contains only the price \( P \) and their dividend function is given by \( f(X, \theta) \) everywhere. The following equation expresses uninformed investors’ conditional expectation of the risky asset’s dividend payoff \( E[V + \nu|P] \):

\[
\int_{\theta}^{0} f(X(P, \theta), \theta) h(\theta|P) \, d\theta = \int_{\theta}^{g(P) - \sigma_s} f(X(P, \theta), \theta) h(\theta|P) \, d\theta + \int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta|P) \, d\theta + \int_{g(P) + \sigma_s}^{\hat{\theta}} f(X(P, \theta), \theta) h(\theta|P) \, d\theta
\]

(25)

where \( h(\theta|P) \) is the density of \( \theta \) conditional on \( P \). The sum of the first and the third terms in Eq. (25) can be written as,

\[
\frac{1}{\bar{\theta} - \hat{\theta}} \left( \int_{\theta}^{g(P) - \sigma_s} f(X(P, \theta), \theta) \, d\theta + \int_{g(P) + \sigma_s}^{\hat{\theta}} f(X(P, \theta), \theta) \, d\theta \right).
\]

Next we consider the second term in Eq. (25). Using Bayes rule we rewrite this term as,

\[
\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta|P) \, d\theta = \text{prob}(g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s)
\]

\[
\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta|P, g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s) \, d\theta.
\]

Let \( \hat{t} = \left( \frac{2\sigma_s}{\sqrt{2\pi}} P + \frac{2\sigma_s}{\sqrt{2}} \hat{M}(P) + g(P) - \sigma_s \right) \). Note that in this region \( \hat{t} = \theta + \left( \frac{2\sigma_s}{\sqrt{2}} \right) g(P) \). Hence, \( \hat{t} \) is a sufficient statistic for the information conveyed by the equilibrium clearing price, \( P \). Thus,

\[
h(\theta|P, g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s) = h(\theta|\hat{t}, g(P) - \sigma_s \leq \theta \leq g(P) + \sigma_s)
\]

\[
= \begin{cases} 
\phi \left( \frac{\theta - \hat{t}}{\sqrt{2\sigma_s^2/2} \sigma_y} \right) \frac{1}{\sqrt{2\pi\sigma_s^2/2}} & \text{if } \theta \in [g(P) - \sigma_s, g(P) + \sigma_s] \\
0 & \text{otherwise}
\end{cases}
\]

\[\text{[27]}\]

The conditional expectation can always be separated into these three regions because \( \hat{t} + \sigma_s \leq g(P) \leq \bar{\theta} - \sigma_s \) for all \( P \).
Thus,
\[
\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) h(\theta | P) \, d\theta = \frac{\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} f(X(P, \theta), \theta) \phi\left(\frac{\theta - \hat{t}}{2\sigma_s \sigma_y / z}\right) \frac{z}{2\sigma_s \sigma_y} \, d\theta}{\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} \phi\left(\frac{\theta - \hat{t}}{2\sigma_s \sigma_y / z}\right) \frac{z}{2\sigma_s \sigma_y} \, d\theta}.
\]

Adding this term to the first and third terms gives the expected dividend payoff of the asset conditional on the price. The conditional variance of the dividend payoff \( Var[V + \nu | P] = \sigma_v^2 + Var[\nu | P] \) where \( Var[\nu | P] \) is given by
\[
\frac{1}{\bar{\theta} - \hat{t}} \left( \int_{\bar{\theta}}^{g(P) - \sigma_s} (f(X(P, \theta), \theta))^2 \, d\theta + \int_{g(P) + \sigma_s}^{\bar{\theta}} (f(X(P, \theta), \theta))^2 \, d\theta \right) - (E[V + \nu | P])^2.
\]

Since the uninformed investors are mean-variance maximizers, their demand \( L(P) \) equals \((w(E[V + \nu | P] - P) / (\rho Var[V + \nu | P]) \) which is a function of \( L(P) \) itself. Thus equilibrium \( L(P) \) is a fixed point of
\[
l = w\frac{E[V + \nu | P, l] - P}{\rho Var[V + \nu | P, l]}.
\]

Next to complete the proof, we show that the fixed point is unique if \( \sigma_v \) is large enough. We will show this in several steps. First, note that \((w(E[V + \nu | P, l] - P) / (\rho Var[V + \nu | P]) \) is bounded so when \( l \) is a very large positive (negative) number it is above (below) the right hand side. More over the right hand side is a continuous function of \( l \), so a fixed point exists. To see that the fixed point is unique, we will show that \((w(E[V + \nu | P, l] - P) / (\rho Var[V + \nu | P]) \) is decreasing in \( l \). To this end, first we show that an increase in \( l \) results in a distribution over \( \theta \) that first order stochastically dominates the earlier one. To see this note that a larger \( l \), leads to a smaller \( \hat{t} \) and changes the distribution only in the range \( \theta \in [g(P) - \sigma_s, g(P) + \sigma_s] \) where it equals \( \phi\left(\frac{\theta - \hat{t}}{2\sigma_s \sigma_y / z}\right) / (\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} \phi\left(\frac{\theta - \hat{t}}{2\sigma_s \sigma_y / z}\right) \frac{z}{2\sigma_s \sigma_y} \, d\theta). \)

We need to show that \( Prob(\theta \leq x) \) is decreasing in \( \hat{t} \). Taking the derivative with respect to \( \hat{t} \) we see that this is true if and only if
\[
- \int_{g(P) - \sigma_s}^{x} \phi' \phi(\theta - \hat{t}) \, d\theta = \frac{\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} \phi(\theta - \hat{t}) \, d\theta}{\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} \phi(\theta - \hat{t}) \, d\theta} > 0.
\]

Plugging \( \phi'(y) = -y \phi(y) \) and rearranging this expression we can rewrite it as
\[
\frac{\int_{g(P) - \sigma_s}^{x} (\theta - \hat{t}) \phi'(\theta - \hat{t}) \, d\theta}{\int_{g(P) - \sigma_s}^{x} \phi(\theta - \hat{t}) \, d\theta} < \frac{\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} (\theta - \hat{t}) \phi(\theta - \hat{t}) \, d\theta}{\int_{g(P) - \sigma_s}^{g(P) + \sigma_s} \phi(\theta - \hat{t}) \, d\theta},
\]

which is clearly true for all \( x \in (g(P) - \sigma_s, g(P) + \sigma_s) \). Therefore, \((\partial E[V + \nu | P, l]) / \partial l < 0 \). To
complete the proof note that the change in \( w(E[V + \nu|P, l] - P)/(\rho Var[V + \nu|P, l]) \) can be written as

\[
\frac{w}{\rho} \frac{\partial E[V + \nu|P, l]}{\partial l} (\sigma_{\nu}^2 + Var[\nu|P, l]) - \frac{\partial Var[\nu|P, l]}{\partial l} ((E[V + \nu|P, l] - P))
\]

which is negative for large enough \( \sigma_{\nu}^2 \). So the fixed point is unique which completes the proof.

**Proof of Lemma 3**

The proof of this lemma is omitted since it is very similar to the proof of Lemma 1.

**Appendix C**

**A Model of \( f(X, \theta) \)**

Suppose that the value of a dot.com firm is determined by \( \pi = \theta + \theta^*e \), where \( \theta \) indicates the quality of the underlying dot.com-related project and is normally distributed with a mean of 0 and a variance of \( \sigma_{\theta}^2 \), and \( e \) is the effort level of the manager. The manager is risk-neutral and is given \( \delta \) share of the firm. She faces a convex cost in exerting efforts, \( \gamma e^2/2 \).

Suppose there exists a derivative market that trades a dot.com index, i.e., \( \theta \). In this market, informed investors receive private signals, \( s = \theta + \sigma_s \epsilon_s \). They are risk-neutral and strategic. The aggregate demand from informed investors is denoted by \( x_i \) and from the noise traders is denoted by \( \sigma_y y \). The total order flow is thus \( X = x_i + \sigma_y y \). The market maker sets the price of the derivative as its expected value conditional on the total order flow, \( X \). The setup of the derivative market is exactly as in Kyle (1989). Hence, we can solve the linear equilibrium following Kyle (1989) where informed investors follow a linear trading rule \( x_i = \beta s \) and the market maker sets the price as \( P = \lambda X \). Then we have

\[ x_i = \beta s ; X = \beta s + \sigma_y y ; P = \lambda X. \]

We first solve for the optimal investment of informed investors. Informed investors choose \( x_i \) to maximize their profit which is

\[
\max_{x_i} x_i E[\theta - P|s] = x_i E \left[ \theta - \frac{\tau_x X}{(\tau_x + \tau_\theta) \beta} \right] s.
\]

The first order condition indicates that \( x_i = \frac{\tau_x}{2\lambda(\tau_\theta + \tau_x)} s \). Hence, \( \beta = \frac{\tau_x}{2\lambda(\tau_\theta + \tau_x)} \). Next we look at the market maker’s inference problem where the market maker will set the price as follows,

\[ P = E[\theta|X] = \frac{\tau_x X}{(\tau_x + \tau_\theta) \beta} = \lambda X, \]
where $\tau_x = \frac{\tau_y \tau_s}{2 \tau_\theta + \tau_s}$. The above equation implies that

$$
\lambda = 2 \sqrt{\left( \frac{\tau_y \tau_s^2}{2 \tau_\theta + \tau_s} - 1 \right) \frac{\tau_y}{\tau_s + \tau_\theta}}.
$$

Finally we solve for the manager’s optimal effort level given that they observe $P$ and $X$. Note that $X$ is a sufficient statistic for $P$ and hence $P$ is redundant in their information set. The manager chooses $e$ to maximize her expected profit which is

$$
\max_e E \left[ \delta \left( \theta + \theta * e - \frac{\gamma e^2}{2} \right) | X \right] - \frac{\tau_x}{\beta(\tau_\theta + \tau_x)} X + \frac{\tau_x}{\beta(\tau_\theta + \tau_x)} X * e - \frac{\gamma e^2}{2}.
$$

The first order condition yields the optimal level of managerial effort as

$$
e = \frac{\tau_x}{\gamma \beta(\tau_\theta + \tau_x)} X.
$$

Hence the dividend received by the shareholders is

$$f(\theta, X) = (1 - \delta)\pi = (1 - \delta) \left( \theta + \frac{\tau_x}{\gamma \beta(\tau_\theta + \tau_x)} X \theta \right).
$$